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On the stability of viscous shock waves

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Abstract

Sufficient conditions for the stability of viscous shock wave solutions to systems of conservation laws are given. The perturbations are assumed to have zero mass and the problem is studied in one space dimension. The analysis applies to non-symmetric constant viscosity matrices and strong shocks. We use Laplace transform techniques and reduce the stability question to a spectral condition for the resolvent equation of the linearized problem.

The spectral condition can be verified analytically in simple cases. In general it must be investigated numerically. We suggest algorithms for this and present numerical results for the case of the Navier-Stokes equations with modified viscosity.

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Contents

Chapter 1

Introduction

1.1 Background

Conservation laws are a class of time-dependent partial differential equations. The fundamental equations in fluid dynamics, Navier-Stokes equations, is a prime example which has been extensively investigated the last hundred years. The governing equations in magnetohydrodynamics is another example of a system of conservation laws.

In many situations it is appropriate to neglect the viscous effects modeled in the conservation law. This leads to simplifications but also to the complication that the solution can have discontinuities. Shock waves are one type of discontinuities. The terminology: shock wave, viscosity, shows the historical dependence on fluid dynamics where shock waves occur in super-sonic flow and viscosity is a physical property of the specific fluid under study.

The role of viscosity in physics and investigation of model problems suggest that the presence of viscosity implies the existence of a continuous, differentiable solution. The mathematical theory does not guarantee this in general. In special cases, for special flows, this can however be proven. This is the situation in the present work where we study solutions "near" a viscous shock wave.

A viscous shock wave is a traveling wave solution with a thin transition region between two sets of values of the variables. The width depends on the viscosity and is smaller for small viscosity. General conditions for the existence of this type of solutions is a subject of research.

Once a stationary or traveling wave solution have been found the question of its stability arises. For physical models only the stable flows are observed in nature, the unstable never occur. This illustrates the importance of the classification of stationary solutions into stable or unstable.

1.2 The present work

In this work we assume the existence of a shock profile and give sufficient conditions for its stability. One line of research in this question is relying on the assumption that the shock is weak. This means that the jump in the variables across the shock is sufficiently small. Adding a natural condition on the non-linearity of the PDE it is then possible to show both existence and stability of so called Lax shocks, see [1],[7],[12]. In a Lax-shock the number of characteristics, of the inviscid problem, entering the shock is $n + 1$ where n is the number of PDE:s, i e the size of the system.

This work is part of another approach where we do not assume that the shock is weak, so the size of the jump can be arbitrary large. Instead we have an assumption on the spectral properties of the problem linearized at the shock profile. Results of this type can be found in [2] and [13]. The present work relies heavily on the methods in [2] and is a generalization of that work. Also we here include computations supporting the theory.

The generalization mentioned concerns the viscosity. In [2] the result is stated under the assumption that the viscosity is the identity matrix, see chapter 2 for the PDE. Here we modify the treatment to allow a constant coefficient matrix with eigenvalues with positive real part. This matrix must also be compatible with the convection part in the sense discussed in [9] in the context of stability of a constant state. Probably the generalization can continue, the next step being a viscosity matrix depending on the space variable. A non-linear viscosity matrix would be necessary to include some physical models.

The main result of the thesis is a stability theorem listing sufficient conditions for stability. One of the conditions must be verified by computations except in the case of a scalar conservation laws. We present computations for Burgers equation and the Navier-Stokes equations for specific shocks, giving numerical results indicating stability. The proof of the main theorem is spread over chapter 2, 3 and 4. In chapter 2 we introduce the conservation law, the stationary solution and the linearized equation. Then we formulate the assumptions and the main theorem. The final part of chapter 2 consists of the last steps in the stability proof. In chapter 3 and 4 we investigate the linearized equation and the resolvent equation which is obtained from the linearized equation by Laplace-transform. In chapter 3 and appendix B we make the preliminary analysis needed in chapter 4 where we derive the crucial estimate of the solutions to the resolvent equation. In chapter 5 we present computations related to the spectral stability condition. We make a finite difference approximation of the eigenvalue problem. Then we apply a shifted inverse iteration to determine the eigenvalues of largest real part of the discrete problem.

1.3 Notation

In this section we introduce the notation for the function spaces and norms we will use. Let $f(x,t)$ be a measurable vector function

$$f : \mathbb{R} \times [0,\infty) \to \mathbb{C}^n.$$

We define the following norms, the derivatives are understood in the sense of distributions.

$$\begin{aligned}
\|f(\cdot,t)\|_{L^m[a,b]} &= \left(\int_a^b |f(x,t)|^m dx\right)^{1/m} \\
\|f(\cdot,t)\|_{W^{m,p}[a,b]} &= \left(\int_a^b \sum_{k=0}^{p} \left|\frac{\partial^k f}{\partial x^k}(x,t)\right|^m dx\right)^{1/m} \\
\|f(\cdot,t)\|_{H^p[a,b]} &= \|f(\cdot,t)\|_{W^{2,p}[a,b]}
\end{aligned}$$

We will only use the cases $m = 1, 2$. When the integral is over the entire real axis we will occasionally not explicitly write the interval.

Chapter 2

Statement of the problem and outline of the proof

The PDE and the stationary solution are introduced in section 2.1. Here we also state the basic assumptions, for example parabolicity of the equation. In the next section we linearize the conservation law at the stationary shock profile and give properties of the linearized equation. In section 2.3 we introduce the more crucial assumptions and define the stability concept we work with. Then we are in a position to state the stability theorem which is the central result of this thesis.

The last part of the stability proof is included in section 2.4. Here we start from the results of the linear analysis derived in chapter 3 and 4. This order of presentation is preferred because the main part of the present work is the investigation of the linearized problem presented in chapter 3 and 4. To use those results to prove linear and non-linear stability then requires comparatively little work, also this part of the proof fits the framework developed in [2].

2.1 The conservation law and the stationary solution.

Consider the partial differential equation

$$v_t + f(v)_x = Bv_{xx}, \qquad (x,t) \in \mathbb{R} \times [0,\infty) \tag{2.1}$$

where

$$\begin{aligned} v :&\quad \mathbb{R} \times [0,\infty) \to \mathbb{R}^n \\ f :&\quad \mathbb{R}^n \to \mathbb{R}^n \end{aligned}$$

and $B \in \mathbb{R}^{n\times n}$. We assume that this PDE has a stationary solution $v(x,t) = U(x)$ tending to constant values for large $|x|$, a shock profile. The limit value for large

x is denoted U_R and that for large negative x, U_L. We have the following assumptions on the conservation law 2.1 and the stationary solution.

Assumption 1 *Equation 2.1 has the following properties:*

(i) The eigenvalues $\{\beta_i\}_{i=1}^n$ *of* B *have positive real part.*

(ii) $f \in C^\infty$.

Assumption 2 *The shock profile* U *has the following properties:*

(i) We have constants K *and* $\beta > 0$ *such that*

$$\begin{cases} |U(x) - U_R| \le Ke^{-\beta x} \\ |U(x) - U_L| \le Ke^{\beta x} \end{cases}$$

(ii) The Jacobian of f *evaluated at* U_R *and* U_L *has real distinct and non-zero eigenvalues. Let* n_R *denote the number of negative eigenvalues of* $f'(U_R)$ *and* n_L *the number of positive eigenvalues of* $f'(U_L)$*, we then have* $n_R + n_L = n + 1$.

The first part of assumption 1 states which type of parabolicity we require of the equation. The smoothness assumption on f is rough, we make no effort to have minimal smoothness assumptions in this thesis. Instead the focus is on structural properties implying stability.

The exponential convergence property of the stationary solution is here stated as an assumption. Actually we can prove that by inserting the ansatz $u(x,t) = U(x)$ in equation 2.1. We then obtain an ODE, which has U_R and U_L as stationary points. Using smoothness of f and properties of the eigenvalues of the matrices $B^{-1}f'(U_R)$ and $B^{-1}f'(U_L)$ proved later we can derive the exponential convergence. Simplicity of presentation is the only reason for stating this as an assumption.

We discuss the second part of assumption 2 using the inviscid equation, by which we means equation 2.1 with $B = 0$. This equation is required to be strictly hyperbolic for $u = U_R$ and U_L respectively. Also we have a Lax shock with $n+1$ characteristics entering the shock.

Finally we remark on the difference if we instead of a stationary solution assume the existence of a traveling wave solution $U(x-\sigma t)$, moving with speed σ. We can then make a coordinate change so that solution is stationary. This leads to a modification of f according to $f_{mod}(u) = f(u) - \sigma u$ and only superficial changes in the treatment.

2.2 The linearized equation

The stability investigation consists of studying the time evolution of small perturbations of the stationary shock profile. Therefore we supplement 2.1 with initial data of the form

$$v(x,0) = U(x) + \epsilon v_{0x}(x) \tag{2.2}$$

where ϵ is small and the perturbation can be written as a derivative because we assume that it has zero mass.

Assumption 3 *The function v_0 and its derivatives belong to L^1 and L^2.*

The linearization consists of deriving the equation satisfied by the perturbation and then neglect the non-linear terms. We will analyze the linearized equation using the Laplace transform. In this context it is convenient to have homogeneous initial data. This leads us to make the substitution

$$v = U + \epsilon v_{0x} e^{-t} + \epsilon u, \tag{2.3}$$

now u will satisfy a problem with homogeneous initial data. The linearization procedure is somewhat lengthy because of the three term ansatz 2.3 and because we need the form of the non-linear terms. Therefore it is placed in appendix A. Here we collect the results. The perturbation u satisfies a PDE of the form

$$u_t + (A(x)u)_x = (Bu_x)_x + h_x(x,t,\epsilon) + \epsilon[g(x,t,u,\epsilon) + H(x,t,\epsilon)u]_x \tag{2.4}$$

where $A = f'(U)$. Below we list properties of the coefficients, forcing and non-linear term in equation 2.4. For details see appendix A.

Since U converges exponentially for large $|x|$, A also has this property:

- There exists constants K_0 and $\beta > 0$ such that

$$\begin{aligned} |A(x) - A_R| &< K_0 e^{-\beta x} \\ |A(x) - A_L| &< K_0 e^{\beta x}. \end{aligned}$$

Because of assumption 3 on v_0 we have the following inequalities.

- The function h and its derivatives belong to $L^1(\mathbb{R} \times [0,\infty))$ and $L^2(\mathbb{R}\times[0,\infty))$. The function H and its derivatives belong to $L^2(\mathbb{R} \times [0,\infty))$. I e we have constants $K_{(p,q)}$ such that

$$\int_0^\infty \left(\| \frac{\partial^q h}{\partial t^q}(\cdot,t)\|_{W^{1,p}} + \| \frac{\partial^q h}{\partial t^q}(\cdot,t)\|^2_{H^p} + \| \frac{\partial^q H}{\partial t^q}(\cdot,t)\|^2_{H^p} \right) dt \le K_{(p,q)}.$$

- For every $c > 0$ there are constants $K'_{(p,q)}$ and $K'_{(p,q,r)}$ such that, for $|u| \le c$, the function g satisfies

$$|\frac{\partial^{p+q} g}{\partial x^p \partial t^q}| \quad \le \quad K'_{(p,q)}|u|^2$$

$$
\begin{aligned}
|\frac{\partial^{p+q+1}g}{\partial x^p \partial t^q \partial u}| &\leq K'_{(p,q)}|u| \\
|\frac{\partial^{p+q+r}g}{\partial x^p \partial t^q \partial u^r}| &\leq K'_{(p,q,r)}
\end{aligned}
$$

2.3 Assumptions and the main theorem

We have introduced the non-linear PDE, the stationary solution and the linearized equation. This is what we need to state the last assumptions which are concerned with the linearized equation. In this subsection we also state the stability theorem.

The constant states to the left and right of the shock must of course be stable. This is ensured by the following assumption on the symbol of the constant coefficient operator of the asymptotic states to the left and right of the shock. This assumption is used in other works see for instance [9].

Assumption 4 *Let $\alpha_i(\xi)$ denote the eigenvalues of the symbol $-i\xi A - \xi^2 B$ where $\xi \in \mathbb{R}$ and $A = A_R$ or A_L. The following inequality holds*

$$\text{Re } \alpha_i(\xi) \leq -\gamma\xi^2$$

for some constant $\gamma > 0$.

To discuss the next assumption we must introduce the eigenvalue problem corresponding to the linearized equation 2.4. A function φ and a number μ are called eigenfunction and eigenvalue respectively, if they satisfy

$$-(A\varphi)_x + B\varphi_{xx} = \mu\varphi \qquad ,\varphi \in L^2. \tag{2.5}$$

As is easily verified, zero is an eigenvalue with eigenfunction U_x. We introduce the notation $\varphi_0 = U_x$. We have the following assumption which is necessary for stability.

Assumption 5 *Zero is the only eigenvalue with* $\text{Re } \mu \geq 0$. *The dimension of the corresponding eigenspace is one.*

The final assumption we need is:

Assumption 6 *The matrix*

$$M = (S_R^{II}\,;\, S_L^I\,;\, U_R - U_L)$$

is non-singular. Here the columns of S_R^{II} are the eigenvectors of $f'(U_R)$ corresponding to positive eigenvalues and S_L^I are the eigenvectors of $f'(U_L)$ corresponding to negative eigenvalues.

This assumption gives additional information on the inhomogeneous version of problem 2.5 for $\mu = 0$. That the matrix M has implications for stability has been used before, see for instance [8].

Our definition as stability is:

Definition 1 *Problem 2.1,2.2 is non-linearly stable under zero mass perturbations if the solution $u(x,t)$ of 2.4 remains smooth for all $t \geq 0$ and $|u(\cdot,t)|$ tends to zero as $t \to \infty$, for ϵ sufficiently small.*

We are now in a position to formulate the main theorem.

Theorem 1 *If assumptions 1-6 are satisfied the problem 2.1, 2.2 is non-linearly stable under zero-mass perturbations.*

2.4 Proof of non-linear stability using resolvent estimates

The linearized equation is equation 2.4 with $\epsilon = 0$:

$$u_t + (Au)_x = Bu_{xx} + h_x \tag{2.6}$$

We recall that the initial data is homogeneous, $u(x,0) = 0$. To investigate 2.6 we introduce the Laplace transform of u

$$\hat{u}(x,s) = \int_0^\infty e^{-st} u(x,t)dt.$$

Applying the Laplace transform to equation 2.6 we obtain

$$s\hat{u} + (A\hat{u})_x = B\hat{u}_{xx} + \hat{h}_x, \tag{2.7}$$

which is called the resolvent equation. In chapter 3 and 4 we make a detailed investigation of 2.7. We show that for Re $s > 0$ there is a unique solution in L^2. This solution satisfies estimates, given below, which are uniform in $|s|$.

The inversion formula for the Laplace transform is

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\eta+i\xi)t} \hat{u}(x, \eta + i\xi) d\xi$$

where $s = \eta + i\xi$ and the estimates on $\hat{u}$ gives the allowed choices of η. Lemma 14 of chapter 4 states that there is a $\delta > 0$ such that

$$\|\hat{u}(\cdot,s)\|_{H^1} \leq K_1 \left[\|\hat{h}(\cdot,s)\|_{L^1} + \|\hat{h}(\cdot,s)\|_{L^2} \right] \tag{2.8}$$

for Re $s > 0$, $|s| \leq \delta$. Lemma 13 of chapter 4 states that

$$\|\hat{u}(\cdot,s)\|_{H^2} \leq K_2 \left[\|\hat{f}(\cdot,s)\|_{L^1} + \|\hat{f}(\cdot,s)\|_{L^2} \right] \tag{2.9}$$

for Re $s > 0$, $|s| \geq \delta$, where $\hat{f} = \hat{h}_x$. Starting from the inequalities 2.8 and 2.9 the treatment is now the same as in [2]. All the differences in this work stemming from the generalization of the viscosity matrix is taken care of in chapter 3 and 4.

We combine 2.8 and 2.9 and use the inverse transform to prove the lemma:

Lemma 1 *The solution of equation 2.6 with homogeneous initial data satisfies the inequality*

$$\int_0^T \left(\|u(\cdot,t)\|_{H^2}^2 + \|u_t(\cdot,t)\|_{L^2}^2\right) dt$$
$$\leq K_3 \left[\left(\int_0^T \|h(\cdot,t)\|_{W^{1,1}} dt\right)^2 + \int_0^T \|h(\cdot,t)\|_{H^1}^2 dt\right]$$

for any $T > 0$. The constant K_3 is independent of T.

Proof: Below C_i denotes constants. We start by differentiating equation 2.7 with respect to x and see that $\hat{u}_x$ satisfies a problem of the same type as that for $\hat{u}$ with the forcing $\hat{h}_x$ replaced by $\hat{h}_{xx} - (A_x\hat{u})_x$. The resolvent estimate 2.8 applied to the problem for $\hat{u}_x$ combined with the original estimate 2.8 give

$$\|\hat{u}(\cdot,s)\|_{H^2} \leq C_1 \left[\|\hat{h}(\cdot,s)\|_{W^{1,1}} + \|\hat{h}(\cdot,s)\|_{H^1}\right] \tag{2.10}$$

for Re $s > 0$, $|s| \leq \delta$. Combining this with 2.9 we see that 2.10 holds for all s with Re $s > 0$. The Parseval equality for the Laplace transform is

$$\int_0^\infty e^{-2\eta t}\|u(\cdot,t)\|_{L^2}^2 dt = \frac{1}{2\pi}\int_{-\infty}^\infty \|\hat{u}(\cdot,\eta+i\xi)\|_{L^2}^2 d\xi.$$

We use the Parseval equality on 2.10 and obtain

$$\int_0^T \|u(\cdot,t)\|_{H^2}^2 dt \leq C_2 \left[\left(\int_0^T \|h(\cdot,t)\|_{W^{1,1}} dt\right)^2 + \int_0^T \|h(\cdot,t)\|_{H^1}^2 dt\right]$$

Here we have left out a few steps, where we use among other things that we can have any $\eta > 0$ in the Parseval equality. We estimate the L^2-norm of u_t by appying the triangle inequality to 2.6. This gives the estimate of the lemma.

The purpose of this section is to start from lemma 1 and prove theorem 1. We will need the following Sobolev inequality

Lemma 2 *Let f be a function of the two variables x and t, defined in the strip $(x,t) \in \mathbb{R} \times [0,T]$. Then we have a constant C such that*

$$\sup_{(x,t)\in\mathbb{R}\times[0,T]} |f(x,t)|^2 \leq C \int_0^T \left(\|f(\cdot,t)\|_{H^2}^2 + \|f_t(\cdot,t)\|_{L^2}^2\right) dt$$

For a proof, see [6]. Linear stability follows from

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}} |u(x,t)| \leq C \lim_{T\to\infty}\int_T^\infty \left(\|u(\cdot,t)\|_{H^2} + \|u_t(\cdot,t)\|_{L^2}\right) dt = 0 \tag{2.11}$$

where we have used lemma 1 with $T = \infty$. To prove non-linear stability ($\epsilon \neq 0$) we will show that the inequality of lemma 1 with $T = \infty$ holds also in this case for sufficiently small ϵ. We will of course need another constant than K_3. We introduce the notation

$$E = K_3 \left[\left(\int_0^\infty \|h(\cdot,t)\|_{W^{1,1}} dt \right)^2 + \int_0^\infty \|h(\cdot,t)\|_{H^1}^2 dt \right].$$

Local existence for the non-linear problem is well-known, so for the non-linear problem we have an estimate of the type in lemma 1 if $T > 0$ is sufficiently small. Now we assume that the estimate breaks down, this means that we have a T_ϵ such that

$$\int_0^{T_\epsilon} (\|u(\cdot,t)\|_{H^2} + \|u_t(\cdot,t)\|_{L^2})\, dt = 4E. \tag{2.12}$$

On the other hand, as long as the solution exists and the norms make sense we can put the $\mathcal{O}(\epsilon)$ terms in the forcing, which leads to

$$\begin{aligned} &\int_0^{T_\epsilon} (\|u(\cdot,t)\|_{H^2} + \|u_t(\cdot,t)\|_{L^2})\, dt \qquad (2.13)\\ &\leq K_1 \left[\left(\int_0^{T_\epsilon} \|h + \epsilon Hu + \epsilon g\|_{W^{1,1}} dt \right)^2 + \int_0^{T_\epsilon} \|h + \epsilon Hu + \epsilon g\|_{H^1}^2 dt \right]. \end{aligned}$$

Now we estimate the terms depending on u in the right hand of 2.13 side in terms of the left hand side. First we of course use the triangle inequality, then we must estimate

$$\|Hu\|_{W^{1,1}}, \quad \|Hu\|_{H^1}, \quad \|g\|_{W^{1,1}} \quad \text{and} \quad \|g\|_{H^1}$$

in terms of $\|u\|_{L^\infty}$ and $\|u\|_{H^1}$, which in turn can be estimated by the left hand side via the Sobolev inequality. It is rather elementary to obtain

$$\begin{aligned} \|Hu\|_{W^{1,1}} &\leq C_4\|u\|_{H^1} \qquad (2.14)\\ \|Hu\|_{H^1} &\leq C_5\|u\|_{H^1}\\ \|g\|_{W^{1,1}} &\leq C_6\|u\|_{H^1}^2\\ \|g\|_{H^1} &\leq C_7\|u\|_{L^\infty}\|u\|_{H^1} \end{aligned}$$

All right hand sides in 2.14 can be estimated in terms of E. Inserting these estimates in 2.13 leads to the inequality

$$4E - \epsilon P(E) \leq 2E \tag{2.15}$$

where $P(E)$ is a polynomial in E whose coefficients can be expressed in C_1, C_2, C_3, C_4 and C. Now we choose $\epsilon = (E/P(E))^{1/2}$, inserting this in 2.15 leads to the contradiction

$$3E \leq 2E.$$

This means that we cannot have equality in 2.12 so the estimate of lemma 1 do not break down. Now non-linear stability follows in the same way as linear stability from equation 2.11.

The derivation here is bit schematic. For a complete treatment we refer to [3], where more details on the inequalities 2.14 are given. An argument concerning the required smoothness of this type of problem is given in [6]. Speaking informally, the solution can lose smoothness only if $\sup_x |u(x,t)|$ becomes large.

Chapter 3

Preliminary analysis of the linearized equation

This and the next chapter consists of a detailed study of the resolvent equation

$$su + (Au)_x = Bu_{xx} + h_x \tag{3.1}$$

which we introduced in section 2.4. We do not use the “hat”-notation here. We introduce L by

$$Lu = Bu_{xx} - (Au)_x$$

and $f = h_x$. So the resolvent equation is written

$$(sI - L)u = f. \tag{3.2}$$

In chapter 4 will determine and estimate the L^2-solution of 3.1 for $\mathrm{Re}\, s > 0$. Here we do a preliminary investigation of 3.1 and the related eigenvalue problem

$$L\phi = \mu\phi, \qquad \phi \in L^2. \tag{3.3}$$

We denote the linear operator defined in 3.3 by $\mathcal{L}$. In section 3.1 we prove a theorem concerning the spectrum of $\mathcal{L}$ which follows from the properties of B only. In section 3.2 we derive several technical lemmas concerning the coefficients of 3.2 for large $|x|$ where they are nearly constant. Then in the last section we are in a position to derive an expression for the solution of 3.2 assuming that the forcing has bounded support.

3.1 A theorem on the eigenvalues of a 1D elliptic operator

We have assumption 5 which is the strong requirement that there are no eigenvalues μ with $\mathrm{Re}\, \mu \geq 0$ and $|\mu| \neq 0$. In this section we prove a theorem on

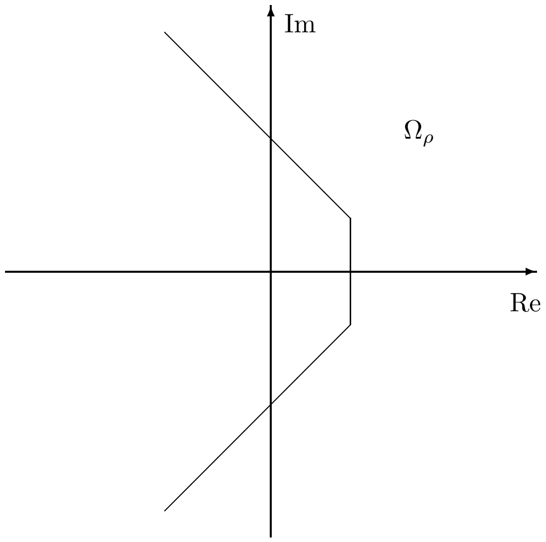

Figure 3.1. The resolvent set of $\mathcal{L}$ contains the region Ω_ρ.

the resolvent set of $\mathcal{L}$, $\rho(\mathcal{L})$. The theorem is a consequence of the fact that $\mathrm{Re}\,\beta_i > 0$, where β_i are the eigenvalues of B. We will below use the scalar product in L^2 with notation

$$(u,v) = \int_{-\infty}^{\infty} u^*(x)v(x)dx$$
$$\|u\|^2 = (u,u)$$

where "$*$" means transpose and complex conjugate. The function space used in this section is exclusively L^2. We also introduce

$$\beta = \min_i \mathrm{Re}\,\beta_i.$$

The proof of the following lemma is found in [6].

Lemma 3 *There exists a matrix* $H = H^* > 0$ *such that*

$$HB + B^*H \geq \frac{\beta}{2}H$$

We now state and prove the theorem saying that the resolvent set contains the region Ω_ρ defined below and shown in figure 3.1.

Theorem 2 *There exists positive constants* K_1, K_2 *and* K_3 *such that the set*

$$\Omega_\rho = \{z \in \mathbb{C} : \mathrm{Re}\, z \geq K_1\} \cup \{z \in \mathbb{C} : |\mathrm{Im} z| \geq K_2 + K_3(K_1 - \mathrm{Re}\, z)\}$$

is contained in the resolvent set, $\Omega_\rho \subset \rho(\mathcal{L})$. *Also for* $s \in \Omega_\rho$ *we have a constant* K_4 *such that*

$$\|u(\cdot,s)\|_{H^2} \leq K_4 \|f(\cdot,s)\|_{L^2}$$

Proof: Below δ_i and C_i will denote positive constants. We multiply the resolvent equation 3.2 with H and then scalar multiply with u, this gives

$$s(u, Hu) = (u, HBu_{xx}) - (u, H(Au)_x) + (u, Hf). \tag{3.4}$$

Next we partially integrate the right hand side and get

$$s(u, Hu) = -(u_x, HBu_x) + f_{rest} \tag{3.5}$$

where $f_{rest} = (u_x, HAu) + (u, Hf)$. The real part of 3.5 is

$$\text{Re } s(u, Hu) = -\frac{1}{2}(u_x, (HB + B^*H)u_x) + \text{Re } f_{rest}. \tag{3.6}$$

The expression (u, Hu) defines a norm equivalent with the L^2-norm, so we have

$$\delta_1 \|v\|^2 \leq (v, Hv) \leq C_1 \|v\|^2$$

for any $v \in L^2$. We now use lemma 3 and simple inequalities in 3.6 to get

$$\text{Re } s(u, Hu) \leq -\frac{\beta}{4}(u_x, Hu_x) + C_2 \left[\sqrt{(u_x, Hu_x)}\sqrt{(u, Hu)} + (u, Hu) + \|f\|^2 \right].$$

The following inequality holds for all $\epsilon > 0$.

$$\sqrt{(u_x, Hu_x)}\sqrt{(u, Hu)} \leq \frac{1}{2\epsilon}(u, Hu) + \frac{\epsilon}{2}(u_x, Hu_x)$$

We choose $\epsilon = \beta/4C_2$ and obtain

$$(\text{Re } s - C_3)(u, Hu) + \frac{\beta}{8}(u_x, Hu_x) \leq C_4 \|f\|^2. \tag{3.7}$$

Using norm equivalence we get

$$(\text{Re } s - C_3)\|u\|^2 \leq C_4 \|f\|^2.$$

Now we choose $K_1 > C_3$ and see that for $\text{Re } s \geq K_1$ we have $\|(sI - \mathcal{L})^{-1}\|$ bounded so we are in the resolvent set. The imaginary part of 3.5 is

$$\text{Im} s(u, Hu) = -\text{Im}(u_x, HBu_x) + \text{Im} f_{rest}.$$

Taking the absolute value we obtain after simple manipulations

$$|\text{Im} s|(u, Hu) \leq C_5 \left[(u_x, Hu_x) + (u, Hu) + \|f\|^2 \right].$$

Rearranging the expression we have

$$(|\text{Im} s| - C_5)(u, Hu) \leq C_5 \left((u_x, Hu_x) + \|f\|^2 \right)$$

which is inserted in 3.7 to give

$$\left[\text{Re } s - K_1 + \frac{\beta}{16}\left(\frac{|\text{Im} s|}{C_5} - 1 \right) \right](u, Hu) + \frac{\beta}{16}(u_x, Hu_x) \leq \left(C_4 + \frac{\beta}{16} \right) \|f\|^2. \tag{3.8}$$

It is now easy to find the constants K_2 and K_3 from the requirement that $\|(sI - \mathcal{L})^{-1}\|$ is bounded.

The inequality 3.8 immediately gives an estimate for $\|u\|_{H^1}$. The L^2-norm of u_{xx} is the estimated by the triangle inequality in 3.1. The proof of the lemma is complete.

3.2 The asymptotic behaviour of the linearized equation for large $|x|$

The resolvent equation 3.2 with $f = 0$ for $x \geq l \gg 1$ has almost constant coefficients. To a first approximation it is of the form

$$su + A_R u_x = B u_{xx}. \tag{3.9}$$

Equation 3.9 written as a system of first order equations is

$$\begin{pmatrix} u \\ v \end{pmatrix}_x = C(s) \begin{pmatrix} u \\ v \end{pmatrix}$$

where

$$C(s) = \begin{pmatrix} B^{-1}A_R & B^{-1} \\ sI & 0 \end{pmatrix}$$

and we have introduced $v = Bu_x - A_R u$. The solutions of 3.9 are polynomials in x multiplied with exponential functions $e^{-\kappa_i(s)x}$ where $\kappa_i(s)$ are determined by the characteristic equation

$$\det(\kappa^2 B - \kappa A_R - sI) = 0.$$

In this section we derive several lemmas concerning the $\kappa_i(s)$ which are eigenvalues of the matrix $C(s)$. Before we start we introduce and recall some notation.

$$T_{1R} A_R S_{1R} = \Lambda_R = \begin{pmatrix} -\Lambda_R^I & 0 \\ 0 & \Lambda_R^{II} \end{pmatrix}$$

where $T_{1R} = S_{1R}^{-1}$, $\Lambda_R^I > 0$ and $\Lambda_R^{II} > 0$. The eigenvalues of A_R are denoted λ_i.

$$T_{2R} B^{-1} A_R S_{2R} = \begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix}$$

where $T_{2R} = S_{2R}^{-1}$. The matrices A^+ and A^- are square, the eigenvalues of A^+ have positive real part and the eigenvalues of A^- have negative real part. That it is possible to block diagonal $B^{-1}A_R$ in this way is shown in this section. The eigenvalues of $B^{-1}A_R$ are denoted τ_i. Finally we have the notation

$$T_{1R} B S_{1R} = \tilde{B}.$$

We start the investigation with the following lemma which gives expressions for $\kappa_i(s)$ for small $|s|$.

Lemma 4 *For small $|s|$ we have the expressions*

$$\begin{aligned}\kappa_i &= \tau_i + O(s^{1/n}) \\ \kappa_{i+n} &= -\frac{s}{\lambda_i} + \frac{s^2 \tilde{b}_{ii}}{\lambda_i^3} + O(s^3)\end{aligned}$$

for $1 \le i \le n$ where $\tilde{b}_{ii}$ are the diagonal elements of $\tilde{B}$.

Proof: The equation for κ_i is a polynomial in κ and s from this it follows that κ_i are continuous functions. Also, if for $s = s_0$, κ_0 is a root of multiplicity p, then we have a constant K such that $|\kappa(s) - \kappa_0| \le K(s - s_0)^{1/p}$. We now solve the characteristic equation for $s = 0$.

$$\det(\kappa^2 B - \kappa A_R) = 0$$

and we get

$$\begin{aligned}\kappa_i(0) &= \tau_i, \\ \kappa_{n+i}(0) &= 0,\end{aligned}$$

for $i = 1, \ldots, n$, so we have the expression for the first n functions κ_i. To find the first non-zero term in the expansion for the n last functions κ_i we make the ansatz $\kappa = \gamma s$ which gives

$$\det(\gamma^2 s^2 B - \gamma s A_R - sI) = 0$$

neglecting the quadratic term we obtain $\kappa_{i+n} \approx -\frac{s}{\lambda_i}$. Now we construct an convergent iteration which gives the s^2 term in the expression for the n last κ_i. We study

$$(\kappa^2 \tilde{B} - \kappa \Lambda_R - sI)x = 0,$$

and make the ansatz

$$\begin{aligned}x &= e_1 + sa \\ \kappa_{n+1} &= -\frac{s}{\lambda_1} + s^2 \beta\end{aligned}$$

where e_1 is the unit vector with the first element equal to one and the rest zero. This ansatz is used for κ_{n+1}, to determine κ_{n+i} we make obvious modifications. The vector a has the first component equal to zero, this takes care of the normalization of the eigenvector, we use the notation

$$a = \begin{pmatrix} 0 \\ \alpha_2 \\ \cdot \\ \alpha_n \end{pmatrix}$$

and have a problem where we will show that the unknowns β, α_2,…,α_n are uniquely determined. Inserting the ansatz and rearranging the expression we arrive at

$$\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \frac{\lambda_2}{\lambda_1}-1 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & \frac{\lambda_n}{\lambda_1}-1 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha_2 \\ \cdot \\ \alpha_n \end{pmatrix} + s\, p(\beta, \alpha_2, \dots, \alpha_n, s) = \begin{pmatrix} \frac{b_{11}}{\lambda_1^2} \\ \frac{b_{12}}{\lambda_1^2} \\ \cdot \\ \frac{b_{1n}}{\lambda_1^2} \end{pmatrix} \tag{3.10}$$

where p is a polynomial in all its arguments. If we solve equation 3.10 for $s = 0$ and obtain

$$\beta = \frac{\tilde{b}_{ii}}{\lambda_i^3}$$

and we have derived the expression in the lemma for $\{\kappa_i(s)\}_{i=n+1}^{2n}$. The iteration on equation 3.10 is now defined by inserting the current iterate of the unknowns in the non-linearity p and solving the linear equation system for the next iterate. Because the function p is Lipschitz continuous in every bounded domain the iteration converges to a solution of equation 3.10 for sufficiently small s, and the proof of the lemma is complete.

The next lemma concerns the sign of the real part of the κ_i for s in the right half plane.

Lemma 5 *For s with $\operatorname{Re} s \geq 0$ and $s \neq 0$, half of the $\kappa_i(s)$ have positive real part and half of them have negative real part.*

Proof: The characteristic equation for large $|s|$

$$\det(\kappa^2 B - sI) \approx 0$$

gives

$$\kappa \approx \pm\sqrt{\frac{s}{\beta_i}}.$$

So the property holds for sufficiently large $|s|$. We now assume that, for some $s = s_0$, a κ_i has real part zero $\kappa_i(s_0) = i\xi_0$ for $\xi_0 \in \mathbb{R} \setminus \{0\}$. We then have

$$\det(-\xi_0^2 B - i\xi_0 A - s_0 I) = 0$$

so s_0 is an eigenvalue of the matrix $-\xi_0^2 B - i\xi_0 A$. According to assumption 4 we then have $\operatorname{Re} s_0 \leq -\delta\xi_0^2$, which is a contradiction. Now, since the κ_i are continuous functions of s they never cross the imaginary axis and the lemma is proven.

The next lemma is now easy to prove.

Lemma 6 *The number of eigenvalues with positive real part of the matrix $B^{-1}A_R$ is the same as the number of positive eigenvalues of A_R.*

Proof: Let r denote the number of positive eigenvalues of A_R. Let r' denote the number of eigenvalues with positive real part of $B^{-1}A_R$. Now we use the expression for κ_i in lemma 4 for $s = \epsilon$ with $0, \epsilon \ll 1$. This gives

$$\begin{aligned} \kappa_i &\approx \tau_i \\ \kappa_{i+n} &\approx -\frac{\epsilon}{\lambda_i} \end{aligned}$$

for $i = 1, \ldots, n$. We see that the number of κ_i with positive real part is $r' + n - r$. Lemma 5 now give $r' = r$.

The following lemma is important for the expansion of the κ_i which are $\mathcal{O}(s)$.

Lemma 7 *The diagonal elements of $\tilde{B}$ have positive real part.*

Proof: Let κ_i be one of the $\mathcal{O}(s)$ size κ_i with positive real part, for $|s| \ll 1$ of course. In the expressions of lemma 4 we see that the corresponding λ_i must be negative for such a κ_i. Now we insert $s = i\epsilon$ with $1 < \epsilon \ll 1$ in the expression and get

$$\text{Re } \kappa_i \approx -\frac{\epsilon^2 \text{Re } \tilde{b}_{ii}}{\lambda_i^3}.$$

The right hand side must be positive and since λ_i is negative, $\tilde{b}_{ii}$ must have positive real part.

Only sign changes are required in the argument to treat a $\tilde{b}_{ii}$ corresponding to a positive λ_i, the proof is complete.

The final result we derive in this section is on how $C(s)$ can be block diagonalized for $|s| \ll 1$.

Lemma 8 *For* Re $s \geq 0$, $0 < |s| \ll 1$ *there exists a matrix $S(s)$ such that*

$$T(s)C(s)S(s) = \begin{pmatrix} C^+(s) & 0 & 0 & 0 \\ 0 & C^-(s) & 0 & 0 \\ 0 & 0 & \mathcal{K}^+(s) & 0 \\ 0 & 0 & 0 & \mathcal{K}^-(s) \end{pmatrix}$$

where $T = S^{-1}$. Here C^+ is a square matrix with eigenvalues with positive real part and C^- is a square matrix with eigenvalues with negative real part. The matrix $\mathcal{K}^+$ is diagonal, the diagonal elements are the $\mathcal{O}(s)$ size κ_i with positive real part. The matrix $\mathcal{K}^-$ is diagonal, the diagonal elements are the $\mathcal{O}(s)$ size κ_i with negative real part. The transformation matrix has the following form

$$S(s) = \begin{pmatrix} S_{2R} & -A_R^{-1}S_{1R} \\ 0 & S_{1R} \end{pmatrix} + \mathcal{O}(s).$$

Proof: We use the expressions of lemma 4 for the κ_i. We see that the $\mathcal{O}(1)$ eigenvalues can be grouped according to the sign of their real part, the $\mathcal{O}(s)$ eigenvalues are distinct for $|s| \ll 1$, Re $s \geq 0$. From these properties the block diagonal form follows by linear algebra.

The form of the transformation matrix is derived by inspection, making the ansatz in powers of s. We first determine an X_1 so that the "(2,1)-block" becomes zero in the transformation:

$$S_1^{-1}\begin{pmatrix} B^{-1}A_R & B^{-1} \\ sI & 0 \end{pmatrix} S_1, \qquad \text{where} \quad S_1 = \begin{pmatrix} I & 0 \\ X_1 & I \end{pmatrix}.$$

Carrying out the multiplication we require that

$$-X_1B^{-1}A_R - X_1B^{-1}X_1 + sI = 0,$$

which gives $X_1 = sA_R^{-1}B + \mathcal{O}(s^2)$. Next we transform to make the "(1,2)-block" zero. I e we determine X_2 so that

$$S_2^{-1}S_1^{-1}CS_1S_2 = \begin{pmatrix} B^{-1}A_R + B^{-1}X_1 & 0 \\ 0 & -X_1B^{-1} \end{pmatrix}$$

where

$$S_2 = \begin{pmatrix} I & X_2 \\ 0 & I \end{pmatrix}.$$

Carrying out the multiplication we require that

$$B^{-1}A_RX_2 + B^{-1}X_1X_2 + B^{-1} + X_2X_1B^{-1} = 0$$

which gives $X_2 = -A_R^{-1} + \mathcal{O}(s)$. Now we have

$$S_2^{-1}S_1^{-1}CS_1S_2 = \begin{pmatrix} B^{-1}A_R + \mathcal{O}(s) & 0 \\ 0 & -sA_R^{-1} + \mathcal{O}(s^2) \end{pmatrix}$$

and we see that a transformation matrix S_3 of the form

$$S_3 = \begin{pmatrix} S_{2R} & 0 \\ 0 & S_{1R} \end{pmatrix}$$

gives $S = S_1S_2S_3$. Calculating the product for $s = 0$ we obtain

$$S(0) = \begin{pmatrix} S_{2R} & -A_R^{-1}S_{1R} \\ 0 & S_{1R} \end{pmatrix}.$$

and the proof is complete.

3.3 Solution of the resolvent equation for forcing with bounded support

We investigate the problem 3.1 in two cases, first for $s = 0$ which is an eigenvalue of the problem 3.3. In this case the solution is not unique, a multiple of the eigenfunction φ_0 can be added. In the second case, with Re $s \geq 0$, $s \neq 0$ we are not at an eigenvalue and the solution is unique. In the first part we use the zero mass assumption, in the second we do not. We derive solution expressions and then use them to obtain estimates of the solution in terms of the forcing.

3.3.1 Solution at the zero eigenvalue

In this section we derive an expression for the solutions of the problem

$$Bu_{xx} - (Au)_x = h_x \qquad , u \in L^2 \tag{3.11}$$

with supp $h \subset [-l_0, l_0]$ for some l_0. Integrating equation 3.11 and multiplying with B^{-1} we arrive at

$$u_x = B^{-1}Au + \tilde{h} \tag{3.12}$$

where the integration constant must be zero because $u \in L^2$. Before we solve 3.12 we discuss the corresponding homogeneous problem

$$\phi_x = B^{-1}A\phi \qquad , \phi \in L^2. \tag{3.13}$$

The coefficient matrix converges exponentially so for large x, 3.13 is of the form

$$\phi_x = \left[B^{-1}A_R + e^{-\beta x}C(x)\right]\phi \tag{3.14}$$

where C is uniformly bounded. By lemma 6 we have a matrix S_{2R} such that

$$T_{2R}B^{-1}A_R S_{2R} = \begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix}$$

where $T_{2R} = S_{2R}^{-1}$. The substitution

$$\phi = S_{2R}\bar{\phi}$$

transforms 3.14 to

$$\bar{\phi}_x = \left[\begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix} + e^{-\beta x}\bar{C}(x)\right]\bar{\phi}. \tag{3.15}$$

Lemma 20 is now applicable. For sufficiently large l we have that equation 3.15 on the interval $[l, \infty)$ together with the condition that $\bar{\phi} \in L^2$ defines a linear mapping of the form

$$\bar{\phi}^I(l) = e^{-\beta l}P(l)\bar{\phi}^{II}(l) \tag{3.16}$$

where P is uniformly bounded. Here $\bar{\phi}^I$ denotes the first r components of $\bar{\phi}$ where r is the size of A^+ and $\bar{\phi}^{II}$ denotes the last $n-r$ components. Conversely, if we choose $\bar{\phi}^{II}(l)$ then equation 3.15 with boundary conditions 3.16 have a unique bounded solution on the interval $[l, \infty)$. Returning to the original variables the boundary conditions are

$$R(l)\phi(l) = 0 \tag{3.17}$$

where we have introduced the notation

$$R(l) = T_{2R}^I - e^{-\beta l}P(l)T_{2R}^{II}.$$

Here T_{2R}^I denotes the r first rows of T_{2R} and T_{2R}^{II} the last $n-r$ rows. Equation 3.13 can be treated in the same way for large negative x and we have boundary conditions

$$R(-l)\phi(-l) = 0. \tag{3.18}$$

The number of equations in 3.17 and 3.18 is determined by the number of eigenvalues with positive real part of $B^{-1}A_R$ and $B^{-1}A_L$ respectively. By lemma 6 these numbers are the same for A_R and A_L. Because we have a Lax shock the total number of equations in 3.17 and 3.18 is $n-1$. We see that we do not have enough boundary conditions to determine the solution uniquely on the interval $[-l, l]$.

We now introduce the concept of fundamental matrices. A fundamental matrix of the ODE 3.13 is a $n \times n$ matrix with columns that are linearly independent solutions of 3.13, we here, of course, include both the bounded and unbounded solutions. We fix a fundamental matrix $\Psi(x)$ and in terms of this $\phi(l)$ and $\phi(-l)$ are related by

$$\phi(l) = \Psi(l)\Psi^{-1}(-l)\phi(-l).$$

We can now summarize the boundary conditions in the relation

$$\begin{bmatrix} R(l)\Psi(l)\Psi^{-1}(-l) \\ R(-l) \end{bmatrix} \phi(-l) = 0 \tag{3.19}$$

According to assumption 5 the solutions of the problem 3.13 have the form $\phi = \alpha\varphi_0$ where α is arbitrary and we recall that $\varphi_0 = U_x$. This gives the following lemma.

Lemma 9 *There exists an l_1 only depending on the matrices B and $A(x)$ such that for $l \geq l_1$ the equation*

$$\begin{bmatrix} R(l)\Psi(l)\Psi^{-1}(-l) \\ R(-l) \end{bmatrix} c = 0$$

implies that $c = \alpha\varphi_0(-l)$ where α is arbitrary.

Now we return to the inhomogeneous problem 3.12. By Duhamel's principle we can write down an expression for the solution

$$u(x) = \Psi(x)u(-l) + \Psi(x)\int_{-l}^{x} \Psi^{-1}(\xi)\tilde{h}(\xi)d\xi \tag{3.20}$$

where it remains to determine $u(-l)$. Just as for the homogeneous problem we can replace the infinite intervals to the left and right by boundary conditions, the cut-off must be done outside the support of $\tilde{h}$. Inserting 3.20 in the boundary conditions we get the following system of linear equations

$$\begin{bmatrix} R(l)\Psi(l)\Psi^{-1}(-l) \\ R(-l) \end{bmatrix} u(-l) = \begin{bmatrix} -R(l)\Psi(l)\int_{-l}^{l} \Psi^{-1}(\xi)\tilde{h}(\xi)d\xi \\ 0 \end{bmatrix}. \tag{3.21}$$

The coefficient matrix in 3.21 is the same as in lemma 9 so $u(-l)$ is not uniquely determined. A side condition must be added. We choose to require that $|u(-l)|$ is minimal. Now $u(-l)$, and therefore u, is determined. This completes the construction of u.

Next we fix l and derive estimates of u in terms of $\tilde{h}$, to begin with on the bounded interval $[-l, l]$. Since $u(-l)$ is determined by minimizing $|u(-l)|$ subject to the constraint 3.21 we have

$$|u(-l)| \leq K_1 \left| \int_{-l}^{l} \Psi^{-1}(\xi)\tilde{h}(\xi)d\xi \right|.$$

Now we use that we have reduced the problem to a bounded interval. The fundamental matrix Ψ is smooth so it and its inverse are uniformly bounded on $[-l, l]$. We get

$$|u(-l)| \leq K_2 \max_{x\in[-l,l]} |\Psi^{-1}(x)| \|\tilde{h}\|_{L^1} \leq K_3 \|\tilde{h}\|_{L^1}.$$

Here and below K_i denote positive constants. We also have $\|\tilde{h}\|_{L^1} \leq \sqrt{2l_0}\|\tilde{h}\|_{L^2}$ so

$$|u(-l)| \leq K_4 \|\tilde{h}\|_{L^2}$$

holds. Again using that $|\Psi|$ is bounded equation 3.20 easily gives

$$\|u\|_{L^\infty[-l,l]} \leq K_5 \|\tilde{h}\|_{L^2}. \tag{3.22}$$

Integrating the absolute value of equation 3.20 we get after simple manipulations

$$\|u\|_{L^1[-l,l]} \leq K_6 \|\tilde{h}\|_{L^2}. \tag{3.23}$$

The Minkowski inequality now gives

$$\|u\|_{L^2[-l,l]} \leq K_7 \|\tilde{h}\|_{L^2}. \tag{3.24}$$

To get estimates of u on the entire real axis we use the boundary conditions 3.17, 3.18 and the now determined $\bar{u}^{II}(l)$ and $\bar{u}^{I}(-l)$ to extend the constructed solution to the intervals $(-\infty, -l]$ and $[l, \infty)$. Lemma 19 is applicable to the extended solution. To the right we have

$$\|u\|_{L^\infty[l,\infty)} + \|u\|_{L^1[l,\infty)} + \|u\|_{L^2[l,\infty)} \leq K_8 |u(-l)| \leq K_9 \|\tilde{h}\|_{L^2} \tag{3.25}$$

and to the left we have the analogous inequality. Combining the inequalities 3.22, 3.23, 3.24 and 3.25 we obtain

$$\|u\|_{L^\infty} + \|u\|_{L^1} + \|u\|_{L^2} \leq K_{10} \|\tilde{h}\|_{L^2}.$$

The last step is now to use the triangle inequality on equation 3.12 to obtain an estimate of the L^2-norm of u_x. We summarize the construction and the estimates in

Lemma 10 *The solutions of the problem 3.12 are given by the expression*

$$u(x) = \alpha\varphi_0 + \Psi(x)u(-l) + \Psi(x) \int_{-l}^{x} \Psi^{-1}(\xi)\tilde{h}(\xi)d\xi$$

where α is arbitrary and $u(-l)$ is determined by minimizing its norm subject to the constraint 3.21. Choosing $\alpha = 0$ we have a solution satisfying

$$\|u\|_{L^\infty} + \|u\|_{L^1} + \|u\|_{H^1} \leq K \|\tilde{h}\|_{L^2}$$

3.3.2 Solution away from the zero eigenvalue

Here we study the problem

$$su + (Au)_x = Bu_{xx} + f \qquad , u \in L^2 \tag{3.26}$$

for s with Re $s \geq 0$, $s \neq 0$. We rewrite equation 3.26 to a system of first order equations by introducing $v = Bu_x - Au$, this gives

$$w_x = C(x,s)w + \tilde{f} \qquad , w \in L^2 \tag{3.27}$$

where $w = (u,v)^T$ and $\tilde{f} = (0, B^{-1}f)^T$. Before we solve 3.27 we study the corresponding homogeneous problem

$$\psi_x = C(x,s)\psi \qquad , \psi \in L^2. \tag{3.28}$$

For large x, equation 3.28 is of the form

$$\psi_x = \left[C(s) + e^{-\beta x} D(x,s)\right] \psi \tag{3.29}$$

where D is uniformly bounded in x for fixed s and analytic in s for fixed x. By lemma 5 there exists a matrix $S(s)$ such that

$$T(s)C(s)S(s) = \begin{pmatrix} \bar{C}^+(s) & 0 \\ 0 & \bar{C}^-(s) \end{pmatrix}$$

where $T = S^{-1}$. The eigenvalues of $\bar{C}^+$ have positive real part and the eigenvalues of $\bar{C}^-$ have negative real part. The substitution

$$\psi(x,s) = S(s)\bar{\psi}(x,s)$$

transforms equation 3.29 to

$$\bar{\psi}_x = \left[\begin{pmatrix} \bar{C}^+(s) & 0 \\ 0 & \bar{C}^-(s) \end{pmatrix} + e^{-\beta x}\bar{D}(x,s).\right]\bar{\psi} \tag{3.30}$$

Lemma 20 is now applicable. For sufficiently large l equation 3.30 on the interval $[l, \infty)$ together with the condition $\bar{\psi} \in L^2$ defines a linear mapping of the form

$$\bar{\psi}^I(l,s) = e^{-\beta l} Q(l,s)\bar{\psi}^{II}(l,s)$$

where Q is uniformly bounded for fixed s. Here $\bar{\psi}^I$ denotes the n first components of $\bar{\psi}$ and $\bar{\psi}^{II}$ the n last. Returning to the original variables the boundary conditions are

$$\Pi(l,s)\psi(l,s) = 0$$

where we have introduced the notation

$$\Pi(l,s) = T^I(s) - e^{-\beta l} Q(l,s) T(s)^{II}.$$

Here T^I denotes the n first rows of T and T^{II} the n last n rows. Equation 3.28 can be treated in the same way for large negative x and we have boundary conditions

$$\Pi(-l,s)\psi(-l,s) = 0.$$

We introduce a fundamental matrix $\Theta(x,s)$ of 3.28. In terms of this $\psi(l,s)$ and $\psi(-l,s)$ are related by

$$\psi(l,s) = \Theta(l,s)\Theta^{-1}(-l,s)\psi(-l,s).$$

We can now summarize the boundary conditions in the relation

$$\begin{bmatrix} \Pi(l,s)\Theta(l,s)\Theta^{-1}(-l,s) \\ \Pi(-l,s) \end{bmatrix} \psi(-l,s) = 0 \tag{3.31}$$

According to assumption 5 we have no eigenvalues for the s considered here, so we have the lemma:

Lemma 11 *For $l \geq l_1$, $\mathrm{Re}\, s \geq 0$, $s \neq 0$ the equation*

$$\begin{bmatrix} \Pi(l,s)\Theta(l,s)\Theta^{-1}(-l,s) \\ \Pi(-l,s) \end{bmatrix} c = 0$$

implies $c = 0$.

Now we return to the inhomogeneous problem 3.27. By Duhamel's principle the solution can be written

$$w(x,s) = \Theta(x,s)\Theta^{-1}(-l,s)w(-l,s) + \Theta(x,s)\int_{-l}^{x} \Theta^{-1}(\xi,s)\tilde{f}(\xi,s)d\xi.$$

Inserting this in the boundary conditions we have

$$\begin{bmatrix} \Pi(l,s)\Theta(l,s)\Theta^{-1}(-l,s) \\ \Pi(-l,s) \end{bmatrix} \psi(-l) = \begin{bmatrix} -\Pi(l)\Theta(l)\int_{-l}^{l} \Theta^{-1}(\xi)\tilde{f}(\xi)d\xi \\ 0 \end{bmatrix} \tag{3.32}$$

To derive estimates uniform in s we restrict s to lie in the compact region $\Omega(\delta)$ given by

$$\Omega(\delta) = \{z \in \mathbb{C} : \mathrm{Re}\, s \geq 0, 0 < \delta \leq |s| \leq K\}.$$

Where K is so large that $\Omega(\delta) \cup \Omega_\rho$ covers the right half plane except for the neighborhood of zero. Here Ω_ρ is the region introduced in section 3.1.

Before we fix l and derive estimates we must discuss how the transformation S, the boundary condition Π and the fundamental matrix Θ depend on s.

For each s we can choose the matrix S to be analytic in a neighborhood of s. Using partition of unity we can paste together a continuous S for the entire region $\Omega(\delta)$. The same remark applies to Θ. Using lemma 21 and the properties of C and D we see that Π analytic in a neighborhood of the s we choose for S and Θ, also Π is continuous in $\Omega(\delta)$.

Now we fix l and the estimates on the bounded interval $[-l, l]$ follows by a similar argument to that given for $s = 0$. Here we use among other things that Θ is bounded for $(x, s) \in [-l, l] \times \Omega(\delta)$ because of compactness. For the extension of the solution to the intervals $(-\infty, -l]$ and $[l, \infty)$ we can again use lemma 19. Here we need that $|\mathrm{Re}\, \kappa_i(s)|$ is uniformly bounded from below for all κ_i by compactness.

We skip the details of the estimate and remark that the treatment is similar to the case $s = 0$. We have the lemma.

Lemma 12 *For $l \geq l_1$, $s \in \Omega(\delta)$ the solution of 3.27 is given by*

$$w(x,s) = \Theta(x,s)\Theta^{-1}(-l,s)w(-l,s) + \Theta(x,s)\int_{-l}^{x} \Theta^{-1}(\xi,s)\tilde{f}(\xi,s)d\xi$$

where $w(-l, s)$ is determined by 3.32. Returning to the original variables of equation 3.26 we have the estimate

$$\|u(\cdot,s)\|_{H^2} \leq K(\delta)\|f(\cdot,s)\|_{L^2}.$$

The expressions derived in this section will be much used in the next chapter. There we will split the solution into several terms, some of which will satisfy problems of the types described here. One of the main points in the next chapter is estimates of the solution in terms of the forcing, then it will be crucial to have the dependence of the estimate on s.

The remarks concerning the analyticity of S, Π and Θ have the following consequence. If we drop the assumption that there are no eigenvalues in the right half plane we still know that there are at most a finite number of eigenvalues and no continuous spectrum in the right half plane. This is important because the eigenvalue condition is to be checked by numerical computation. That we only have a point spectrum follows from equation 3.31. The eigenvalues are the zeros of the determinant of the coefficient matrix and we have the freedom of choice to have an analytic determinant in a neghborhood of any s. Thus if s is an eigenvalue there are no eigenvalues in a neighborhood of s, because the zeros of an analytic function are isolated. In addition according to section 3.1 there are no eigenvalues for sufficiently large $|s|$ in the right half plane, so if they exist they must be located in a bounded region and there can be no accumulation points.

Chapter 4

Estimates for the solution of the resolvent equation

In this chapter we study the resolvent equation

$$s\hat{u} + (A(x)\hat{u})_x = B\hat{u}_{xx} + \hat{h}_x, \tag{4.1}$$

for $s \in \mathbb{C}$, which is obtained by Laplace-transform of equation 2.4 with $\epsilon = 0$. We show below that 4.1 has a unique solution in L^2 for Re $s > 0$. We will also derive estimates for the solution.

The investigation is divided in two parts. The more standard part, when $|s|$ is bounded away from zero, is presented in section 4.1. In section 4.2 we turn to the case $|s| \ll 1$, Re $s > 0$. Here two difficulties arise. First we are near the eigenvalue zero and second the decay rate in space deteriorates. Both of these facts complicates the derivation of the required estimate. In the second part the assumption of zero mass of the forcing is necessary.

4.1 Values of the time conjugate variable bounded away from the zero eigenvalue

The resolvent equation is studied for Re $s \geq 0$, $|s| \geq \delta > 0$. The results of this section are summarized in the lemma:

Lemma 13 *For* Re $s \geq 0$, $|s| \geq \delta > 0$, *there is a unique solution of 4.1 in* L^2. *This solution satisfies*

$$\|\hat{u}(\cdot, s)\|_{H^2} \leq K(\delta) \left(\|\hat{f}(\cdot, s)\|_{L^1} + \|\hat{f}(\cdot, s)\|_{L^2} \right)$$

where $\hat{f} = \hat{h}_x$.

According to lemma 1 in section 3.1 the inequality holds for Ω_ρ. It remains to show the inequality for $s \in \Omega(\delta)$ which is the compact region defined in section 3.3.

We drop the "hat"-notation in equation 4.1 and introduce the variable $v = Bu_x - Au$ satisfying $v_x = su - f$. This gives the system of first order equations

$$\begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} B^{-1}A & B^{-1} \\ sI & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + g$$

where $g = (0, -B^{-1}f)^T$. We will now split $(u, v)^T$ in several parts which satisfy problems of the types treated in section 3.3 and appendix B. We start by introducing a function $(u_R, v_R)^T$ satisfying

$$\begin{pmatrix} u_R \\ v_R \end{pmatrix}_x = \left[C(s) + e^{-\beta x} D(x, s)\right] \begin{pmatrix} u_R \\ v_R \end{pmatrix} + g, \qquad x \in [l-1, \infty) \tag{4.2}$$

where C and D were introduced in chapter 3 and l is to be fixed later. We make the substitution

$$\begin{pmatrix} u_R \\ v_R \end{pmatrix} = S(s) \begin{pmatrix} \bar{u}_R \\ \bar{v}_R \end{pmatrix}$$

with S as in section 3.3. This gives

$$\begin{pmatrix} \bar{u}_R \\ \bar{v}_R \end{pmatrix}_x = \left[\begin{pmatrix} \bar{C}^{+}(s) & 0 \\ 0 & \bar{C}^{-}(s) \end{pmatrix} + e^{-\beta x} \bar{D}(x, s)\right] \begin{pmatrix} \bar{u}_R \\ \bar{v}_R \end{pmatrix} + \bar{g}.$$

We now choose homogeneous boundary conditions

$$\bar{u}_R(l-1) = 0$$

and use lemma 19. For sufficiently large l the solution is uniquely determined and we have estimates

$$\begin{aligned} &\|v_R\|_{L^\infty[l-1,\infty)} + \|u_R\|_{L^\infty[l-1,\infty)} + \|u_R\|_{H^1[l-1,\infty)} \\ &\quad \le K_1(\delta) \left[\|f\|_{L^1[l-1,\infty)} + \|f\|_{L^2[l-1,\infty)}\right] \end{aligned} \tag{4.3}$$

where we have returned to the original variables and have used equation 4.2 to obtain the L^2-estimate for u_{Rx}. The variable change only changes the constant in the estimate. The estimate is uniform in s since S is continuous and $\Omega(\delta)$ is compact.

On the interval $(-\infty, -l+1]$ we introduce a completely analogous problem with unknown $(u_L, v_L)^T$. These functions satisfy an estimate of the type 4.3 on the interval $(-\infty, -l+1]$. Now we introduce monotone, smooth cut-off functions φ_L and φ_R where

$$\varphi_R(x) = \begin{cases} 1, & x \in [l, \infty) \\ 0, & x \in (-\infty, l-1] \end{cases}$$

and $\varphi_L(x) = \varphi_R(-x)$. Using the cut-off functions and the solutions of the left and right problems we introduce the function $(u_M, v_M)^T$ by the relation

$$\begin{pmatrix} u_M \\ v_M \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} - \varphi_R \begin{pmatrix} u_R \\ v_R \end{pmatrix} - \varphi_L \begin{pmatrix} u_L \\ v_L \end{pmatrix}. \tag{4.4}$$

The new function now satisfy the equation

$$\begin{pmatrix} u_M \\ v_M \end{pmatrix}_x = \begin{pmatrix} B^{-1}A & B^{-1} \\ sI & 0 \end{pmatrix} \begin{pmatrix} u_M \\ v_M \end{pmatrix} + \tilde{g}$$

where

$$\tilde{g} = g(1 - \varphi_R - \varphi_L) - \varphi_R' u_R - \varphi_L' u_L$$

and thus have bounded support. This problem is of the type treated in lemma 12. We have the estimate

$$\|u_M\|_{H^1} \leq K_2(\delta)\|\tilde{g}\|_{L^2}. \tag{4.5}$$

The norm of $\tilde{g}$ can be estimated by the corresponding norms for g and the L^∞-norms of u_R and u_L. We combine the inequalities 4.3 and 4.5 and use the resolvent equation to obtain an L^2-estimate of u_{xx}. This gives the estimate of lemma 13

4.2 Small values of the time conjugate variable

According to the previous section we have the estimate of lemma 13 on any region Re $s \geq 0$ which excludes a neighborhood of the zero eigenvalue. In this section we use the zero mass assumption to derive the estimate for Re $s \geq 0$, $0 < |s| \ll 1$. Part of the proof consists of an iteration which converges for sufficiently small $|s|$. This means that the methods in this section give the estimate on the region $\{z \in \mathbb{C} : \text{Re } z \geq 0\} \setminus \Omega(\delta)$. This makes it possible to combine the results of lemma 13 and 14 which we do in section 2.4.

More precisely the result of this section is:

Lemma 14 *It exists a δ such that for* Re $s \geq 0$, $0 < |s| \leq \delta$ *the L^2-solution of 4.1 satisfies the inequality*

$$\|\hat{u}(\cdot, s)\|_{H^1} \leq K \left(\|\hat{h}(\cdot, s)\|_{L^1} + \|\hat{h}(\cdot, s)\|_{L^2} \right).$$

The proof of this estimate is the content of the rest of the chapter. Below we drop the "hat"-notation.

4.2.1 Reducing the forcing to $\mathcal{O}(s)$

In this section we construct a solution to the problem

$$u_{1x} = B^{-1}Au_1 + \tilde{h}, \qquad u_1 \in L^2. \tag{4.6}$$

where $\tilde{h} = -B^{-1}h$. This function will then be related to the original unknown by $u = u_1 + u_2$, where u_2 solves an equation of the type 4.1 with forcing of size $\mathcal{O}(s)$. As we will see in the construction the solution of 4.6 is not unique. The arbitrariness in the definition of u_1 will of course be reflected in the equation for u_2 since u is uniquely defined by the condition that it is in L^2.

We start by introducing u_{1R} satisfying

$$u_{1Rx} = B^{-1}Au_{1R} + \tilde{h}, \qquad x \in [l-1, \infty) \tag{4.7}$$

where l remains to be chosen. We make the substitution

$$u_{1R} = S_{2R}\bar{u}_{1R}.$$

This transforms equation 4.7 to

$$\bar{u}_{1R} = \left[\begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix} + e^{-\beta x}\bar{C}(x) \right] \bar{u}_{1R} + \bar{h}, \qquad x \in [l-1, \infty) \tag{4.8}$$

where A^+ and A^- were introduced in section 3.2. We now choose homogeneous boundary conditions

$$\bar{u}^I_{1R}(l) = 0 \tag{4.9}$$

where $\bar{u}^I_{1R}$ denotes the r first components of $\bar{u}_{1R}$ and r is the size of A^+. The problem 4.8, 4.9 is of the type treated in lemma 19, consequently for sufficiently large l we have a unique solution satisfying

$$\|u_{1R}\|_{L^\infty[l-1,\infty)} + \|u_{1R}\|_{L^1[l-1,\infty)} + \|u_{1R}\|_{L^2[l-1,\infty)} \leq K_1 \|\tilde{h}\|_{L^1[l-1,\infty)}$$

where we have usedreturned to the original variables. Using the triangle inequality in equation 4.7 we can also get an estimate of the L^2-norm of u_{1Rx}. Summarizing, we have

$$\begin{aligned} &\|u_{1R}\|_{L^\infty[l-1,\infty)} + \|u_{1R}\|_{L^1[l-1,\infty)} + \|u_{1R}\|_{H^1[l-1,\infty)} \\ &\quad \leq K_2 \left(\|\tilde{h}\|_{L^1[l-1,\infty)} + \|\tilde{h}\|_{L^2[l-1,\infty)} \right). \end{aligned} \tag{4.10}$$

In complete analogy with the above we introduce a function u_{1L} defined on the interval $(-\infty, -l+1]$. For u_{1L} we introduce homogeneous boundary conditions at $a = -l + 1$ and we arrive at an estimate of the type 4.10.

Now we introduce a function u_{1M} and relate it to the constructed functions u_{1R}, u_{1L} and to the original function u_1 by the relation

$$u_1 = u_{1M} + \varphi_R u_{1R} + \varphi_L u_{1L} \tag{4.11}$$

where φ_R and φ_L are the cut-off functions introduced in section 4.1. The function u_{1M} satisfies the following equation

$$u_{1Mx} = B^{-1} A u_{1M} + h_M$$

where the forcing is given by

$$h_M = \tilde{h}(1 - \varphi_R - \varphi_L) - \varphi_R' u_R - \varphi_L' u_L, \tag{4.12}$$

and we see that supp $h_M \subset [-l, l]$. This problem is of the type treated in section 3.3. According to lemma 10 we can construct a solution satisfying

$$\|u_{1M}\|_{L^1} + \|u_{1M}\|_{H^1} \leq K_3 \|h_M\|_{L^2}. \tag{4.13}$$

The following estimate is easily obtained using equation 4.12 and the L^∞-estimate of 4.10.

$$\|h_M\|_{L^2} \leq K_4 \left(\|\tilde{h}\|_{L^1} + \|\tilde{h}\|_{L^2} \right) \tag{4.14}$$

Combining the inequalities 4.10, 4.13 and 4.14 in the expression 4.11 for u_1 we have proven:

Lemma 15 *We can construct a solution u_1 of the problem 4.7 satisfying*

$$\|u_1\|_{L^1} + \|u_1\|_{H^1} \leq K \left(\|h\|_{L^1} + \|h\|_{L^2} \right)$$

where $h = B\tilde{h}$.

4.2.2 Obtaining bounded support for the forcing

The splitting $u = u_1 + u_2$ gives the following equation for u_2.

$$s u_2 + (A u_2)_x = B u_{2xx} - s u_1 \tag{4.15}$$

We introduce $v_2 = B u_{2x} - A u_2$ and rewrite equation 4.15 to a system of first order equations

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}_x = \begin{pmatrix} B^{-1}A & B^{-1} \\ sI & 0 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} + s f_2 \tag{4.16}$$

where $f_2 = (0, u_1)^T$. By lemma 15 we have estimates of f_2 in terms of the original forcing h. In this subsection we will construct left and right solutions to get a problem with bounded support of the forcing. A complication in this case is that the real part of the eigenvalues $\kappa_i(s)$ is not bounded away from zero because $|s|$ can be arbitrarily small. The decay rate in space deteriorates. Since the forcing now is proportional to s it is still possible to get an s-independent estimate. The results of the construction are summarized in lemma 16 below.

We introduce a function $(u_{2R}, v_{2R})^T$ satisfying

$$\begin{pmatrix} u_{2R} \\ v_{2R} \end{pmatrix}_x = \left[C(s) + e^{-\beta x} D(x,s)\right] \begin{pmatrix} u_{2R} \\ v_{2R} \end{pmatrix} + s f_2, \qquad x \in [l-1, \infty) \quad (4.17)$$

where l remains to be chosen. We make the substitution

$$\begin{pmatrix} u_{2R} \\ v_{2R} \end{pmatrix} = S(s) \begin{pmatrix} \bar{u}_{2R} \\ \bar{v}_{2R} \end{pmatrix}$$

where S was introduced in lemma 8. This transforms 4.17 to

$$\begin{pmatrix} \bar{u}_{2R} \\ \bar{v}_{2R} \end{pmatrix}_x = \left[\begin{pmatrix} C^+ & 0 & 0 & 0 \\ 0 & C^- & 0 & 0 \\ 0 & 0 & \mathcal{K}^+ & 0 \\ 0 & 0 & 0 & \mathcal{K}^- \end{pmatrix} + e^{-\beta x} \bar{D} \right] \begin{pmatrix} \bar{u}_{2R} \\ \bar{v}_{2R} \end{pmatrix} + s \bar{f}_2,$$

see lemma 8 for the properties of the coefficient matrix. The matrix C^+ has size r_1 and $\mathcal{K}^+$ has size r_2. We choose homogeneous boundary conditions

$$\begin{cases} \bar{u}^I_{2R}(l-1) = 0 \\ \bar{v}^I_{2R}(l-1) = 0 \end{cases}$$

where $\bar{u}^I_{2R}$ denotes the first r_1 components of $\bar{u}_{2R}$ and $\bar{v}^I_{2R}$ denotes the first r_2 components of $\bar{v}^I_{2R}$. We are now in a position to apply lemma 19. We obtain the following

$$\|u_{2R}\|_{L^\infty[l-1,\infty)} + \|v_{2R}\|_{L^\infty[l-1,\infty)} \leq |s| K_1 \|f_2\|_{L^1[l-1,\infty)}$$

where we have returned to the original variables. The change of variables does not affect the estimate, see lemma 8. For the L^2-estimates it is crucial how the real part of the κ_i depend on s. Combining lemma 7 and lemma 4 we obtain the inequality

$$|\text{Re}\ \kappa_i(s)| \geq \delta_1 |s|^2 \qquad (4.18)$$

for $n+1 \leq i \leq 2n$ and some $\delta_1 > 0$. We recall that the first n of the κ_i have real part of size $\mathcal{O}(1)$. We now apply lemma 19 and use 4.18 to get the estimate

$$\|u_{2R}\|_{L^2[l-1,\infty)} \leq \frac{K_2}{\sqrt{\min_i |\text{Re}\ \kappa_i(s)|}} \|sf\|_{L^1[l-1,\infty)} \leq \frac{K_3}{|s|} |s| \|f\|_{L^1[l-1,\infty)}.$$

The final estimate we need is for the L^2 norm of u_{2Rx}. This is obtained by the triangle inequality in 4.16, using that the coefficient matrix is uniformly bounded. Summarizing the estimates we have the lemma:

Lemma 16 *The solution of 4.15 with n homogeneous boundary conditions at* $x = l - 1$ *satisfy*

$$\begin{array}{rcl} \|u_{2R}\|_{H^1[l-1,\infty)} & \leq & K\|f_2\|_{L^1[l-1,\infty)} \\ \|u_{2R}\|_{L^\infty[l-1,\infty)} + \|v_{2R}\|_{L^\infty[l-1,\infty)} & \leq & |s|K\|f_2\|_{L^1[l-1,\infty)}. \end{array}$$

In the same way we can construct $(u_{2L}, v_{2L})^T$ on the interval $(-\infty, -l+1]$ and obtain estimates of the type in lemma 16.

4.2.3 Determining the component in the zero eigenspace

We start this section by introducing a new function $(u_{2M}, v_{2M})^T$ and relating the functions $(u_{2L}, v_{2L})^T$ and $(u_{2R}, v_{2R})^T$ to the function $(u_2, v_2)^T$ by

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_{2M} \\ v_{2M} \end{pmatrix} + \varphi_R \begin{pmatrix} u_{2R} \\ v_{2R} \end{pmatrix} + \varphi_L \begin{pmatrix} u_{2L} \\ v_{2L} \end{pmatrix}. \tag{4.19}$$

The introduced function satisfy the equation

$$\begin{pmatrix} u_{2M} \\ v_{2M} \end{pmatrix}_x = \begin{pmatrix} B^{-1}A & B^{-1} \\ sI & 0 \end{pmatrix} \begin{pmatrix} u_{2M} \\ v_{2M} \end{pmatrix} + s \begin{pmatrix} g_M \\ h_M \end{pmatrix} \tag{4.20}$$

where the forcing is given by

$$s \begin{pmatrix} g_M \\ h_M \end{pmatrix} = s(1 - \varphi_R - \varphi_L) f_2 - \varphi_R' \begin{pmatrix} u_{2R} \\ v_{2R} \end{pmatrix} - \varphi_L' \begin{pmatrix} u_{2L} \\ v_{2L} \end{pmatrix} \tag{4.21}$$

and have bounded support contained in $[-l_0, l_0]$ for some l_0. Via lemma 15 and 16 we can estimate (g_M, h_M) in terms of the original forcing h.

We treated a problem similar to 4.20 in section 4.1. The difference here is that we cannot restrict s to lie in a compact region since we must exclude $s = 0$. This makes it impossible to employ the methods of section 4.1. There we, for example, relied on the fact that the fundamental matrix is uniformly bounded for $(x, s) \in [-l, l] \times \Omega(\delta)$ since it is continuous and the domain is compact.

Here we will be forced to split the unknown further to obtain the estimate. First we derive boundary conditions at $x = \pm l$, for $l \geq l_0$, which replace the equation on the intervals $(-\infty, -l]$ and $[l, \infty)$. The expressions we derive will be rather unwieldy because it is necessary to express the asymptotic behaviour of the boundary conditions for large l and and small $|s|$.

On the interval to the right, equation 4.20 is of the form

$$\begin{pmatrix} u_{2M} \\ v_{2M} \end{pmatrix}_x = \left[C(s) + e^{-\beta x} D(x, s)\right] \begin{pmatrix} u_{2M} \\ v_{2M} \end{pmatrix}, \qquad x \in [l, \infty).$$

We make the substitution

$$\begin{pmatrix} u_{2M} \\ v_{2M} \end{pmatrix} = S(s) \begin{pmatrix} \bar{u}_{2M} \\ \bar{v}_{2M} \end{pmatrix}$$

which leads to

$$\begin{pmatrix} \bar{u}_{2M} \\ \bar{v}_{2M} \end{pmatrix}_x = \left[\bar{C}(s) + e^{-\beta x}\bar{D}(x,s)\right] \begin{pmatrix} \bar{u}_{2M} \\ \bar{v}_{2M} \end{pmatrix}, \qquad x \in [l, \infty) \tag{4.22}$$

where the properties of S, C and $\bar{C}$ are given in lemma 8. The matrix $\bar{D}$ is of the form

$$\bar{D}(x,s) = \begin{pmatrix} \bar{D}_{11}(x) & \bar{D}_{12}(x) \\ 0 & 0 \end{pmatrix} + s\bar{D}_1(x,s)$$

where the matrices $\bar{D}_{11}$, $\bar{D}_{12}$ and $\bar{D}_1$ are uniformly bounded. Equation 4.22 is of the form treated in lemma 20 and because of the special form of $\bar{D}$ we have the boundary conditions

$$\begin{pmatrix} \bar{u}^I_{2M}(l) \\ \bar{v}^I_{2M}(l) \end{pmatrix} = e^{-\beta l} Q(l,s) \begin{pmatrix} \bar{u}^{II}_{2M}(l) \\ \bar{v}^{II}_{2M}(l) \end{pmatrix} \tag{4.23}$$

where Q has the form

$$Q(l,s) = \begin{pmatrix} P(l) & Q_{12}(l) \\ 0 & 0 \end{pmatrix} + s \begin{pmatrix} Q_1(l,s) \\ Q_2(l,s) \end{pmatrix}.$$

Here P was introduced in section 3.3 and Q_{12}, Q_1 and Q_2 are uniformly bounded. Boundary conditions to the left are introduced in the same manner:

$$\begin{pmatrix} \bar{u}^{II}_{2M}(-l) \\ \bar{v}^{II}_{2M}(-l) \end{pmatrix} = e^{-\beta l} Q(-l,s) \begin{pmatrix} \bar{u}^I_{2M}(-l) \\ \bar{v}^I_{2M}(-l) \end{pmatrix} \tag{4.24}$$

where

$$Q(-l,s) = \begin{pmatrix} P(-l) & Q_{12}(-l) \\ 0 & 0 \end{pmatrix} + s \begin{pmatrix} Q_1(-l,s) \\ Q_2(-l,s) \end{pmatrix}$$

Now we have introduced the equation for $(u_{2M}, v_{2M})^T$ and the cut-off boundary conditions. The last splitting is now done according to

$$\begin{pmatrix} u_{2M} \\ v_{2M} \end{pmatrix} = \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} + \begin{pmatrix} \alpha\varphi_0 \\ 0 \end{pmatrix} + \begin{pmatrix} u_4 \\ v_4 \end{pmatrix}$$

The introduced functions $(u_3, v_3)^T$ and the number α satisfies the problem

$$\begin{pmatrix} u_3 \\ v_3 \end{pmatrix}_x = \begin{pmatrix} B^{-1}A & B^{-1} \\ sI & 0 \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} + s \begin{pmatrix} g_M \\ h_M - \alpha\varphi_0 \end{pmatrix}, \qquad x \in [-l, l] \tag{4.25}$$

with the boundary conditions 4.23, 4.24. The idea behind the splitting is the observation that $u_{2M} \approx \alpha\varphi_0$ for small $|s|$ since we are near the zero eigenvalue. We therefore construct u_3, the $\mathcal{O}(s)$ deviation from $\alpha\varphi_0$, by subtracting $\alpha\varphi_0$ from the forcing in the proper place. When we construct u_3 we work on the bounded interval, this leads to the $\mathcal{O}(e^{-\beta l})$ remainder u_4. This was a very schematic description of what we will do in detail below.

The solution $(u_3, v_3)^T$, α will be constructed by an iteration. The first iterate for $v_3^{(1)}$ and $\alpha^{(1)}$ solves the problem

$$\begin{cases} v_{3x}^{(1)} = sh_M - s\alpha^{(1)}\varphi_0 \\ \bar{v}_3^{(1)I}(l) = 0 \\ \bar{v}_3^{(1)II}(-l) = 0 \end{cases}$$

where the boundary conditions come from 4.23, 4.24 with $s = 0$. After $v_3^{(1)}$ and $\alpha^{(1)}$ have been determined we solve the following problem for $u_3^{(1)}$

$$\begin{cases} u_{3x}^{(1)} = B^{-1}Au_3^{(1)} + B^{-1}v_3^{(1)} + sg_M \\ \bar{u}_3^{(1)I}(l) = e^{-\beta l}P(l)\bar{u}_3^{(1)II}(l) + e^{-\beta l}Q_{12}(l)\bar{v}_3^{(1)II}(l) \\ \bar{u}_3^{(1)II}(-l) = e^{-\beta l}P(-l)\bar{u}_3^{(1)I}(-l) + e^{-\beta l}Q_{12}(-l)\bar{v}_3^{(1)I}(-l). \end{cases} \tag{4.26}$$

The same remark concerning $s = 0$ in the boundary conditions apply here. For the i:th iterate $v_3^{(i)}$, $\alpha^{(i)}$ we have

$$\begin{cases} v_{3x}^{(i)} = su_3^{(i-1)} - s\alpha^{(i)}\varphi_0 \\ \bar{v}_3^{(i)I}(l) = se^{-\beta l}Q_2(l,s)\begin{pmatrix} \bar{u}_3^{(i-1)II}(l) \\ \bar{v}_3^{(i-1)II}(l) \end{pmatrix} \\ \bar{v}_3^{(i)II}(-l) = se^{-\beta l}Q_2(-l,s)\begin{pmatrix} \bar{u}_3^{(i-1)I}(-l) \\ \bar{v}_3^{(i-1)I}(-l) \end{pmatrix} \end{cases}$$

We see that the $\mathcal{O}(s)$ deviation of the boundary conditions 4.23, 4.24 from the case $s = 0$ leads, in the iteration to inhomogeneous boundary conditions. We will in the investigation use that this inhomogeneity is $\mathcal{O}(s)$ and can be estimated in terms of the previous iterate. For the i:th iterate $u_3^{(i)}$ we have

$$\begin{cases} u_{3x}^{(i)} = B^{-1}Au_3^{(i)} + B^{-1}v_3^{(i)} \\ \bar{u}_3^{(i)I}(l) = e^{-\beta l}P(l)\bar{u}_3^{(i)II}(l) + e^{-\beta l}Q_{12}(l)\bar{v}_3^{(i)II}(l) + se^{-\beta l}Q_1(l,s)\begin{pmatrix} \bar{u}_3^{(i-1)II}(l) \\ \bar{v}_3^{(i-1)II}(l) \end{pmatrix} \\ \bar{u}_3^{(i)II}(-l) = e^{-\beta l}P(-l)\bar{u}_3^{(i)I}(-l) + e^{-\beta l}Q_{12}(-l)\bar{v}_3^{(i)I}(-l) \\ \qquad +se^{-\beta l}Q_1(-l,s)\begin{pmatrix} \bar{u}_3^{(i-1)I}(-l) \\ \bar{v}_3^{(i-1)I}(-l) \end{pmatrix} \end{cases}$$

Here we also see that the $\mathcal{O}(s)$ deviation is treated as an inhomogeneity proportional to the previous estimate.

We now determine $v_3^{(1)}$ and $\alpha^{(1)}$ The boundary conditions give

$$\begin{cases} v_3^{(1)}(l) = \left[S_{1R}^{II} + \mathcal{O}(s)\right]\bar{v}_3^{(1)II}(l) \\ v_3^{(1)}(-l) = \left[S_{1L}^{I} + \mathcal{O}(s)\right]\bar{v}_3^{(1)I}(-l). \end{cases} \tag{4.27}$$

Integrating the equation for $v_3^{(1)}$ we obtain

$$v_3^{(1)}(l) - v_3^{(1)}(-l) = s\int_{-l}^{l} h_M(x)dx - s\alpha^{(1)}\int_{-l}^{l}\varphi_0(x)dx \tag{4.28}$$

The equations 4.27 and 4.28 is a system of linear equations for $\bar v_3^{(1)II}(l)$, $\bar v_3^{(1)I}(-l)$ and $\alpha^{(1)}$ which can be rewritten to

$$\left(S_{2R}^{II} + \mathcal{O}(s)\,;\; -S_{1L}^{I} + \mathcal{O}(s)\,;\; \int_{-l}^{l}\varphi_0 dx \right) \begin{pmatrix} \bar v_3^{(1)II}(l) \\ \bar v_3^{(1)I}(-l) \\ s\alpha^{(1)} \end{pmatrix} = s\int_{-l}^{l} h_M dx \tag{4.29}$$

Since φ_0 is the derivative of the shock profile we have

$$\int_{-l}^{l}\varphi_0 dx = U_R - U_L + \mathcal{O}(e^{-\beta l}).$$

Assumption 6 now implies that 4.29 is solvable for l sufficiently large and $|s|$ sufficiently small. Now $\bar v_3^{(1)}(l)$, $\bar v_3^{(1)}(-l)$, $\alpha^{(1)}$ and therefore $v_3^{(1)}(x)$ are determined. Since $v_3^{(1)}(x)$ is expressed using the integral of h_M it is easy to obtain the estimates

$$\begin{aligned} \|v_3^{(1)}\|_{L^\infty[-l,l]} + \|v_3^{(1)}\|_{H^1[-l,l]} &\leq sK(l)\|h_M\|_{L^2} \\ |\alpha^{(1)}| &\leq K(l)\|h_M\|_{L^1} \end{aligned} \tag{4.30}$$

where K is a polynomial in l.

Next we turn to the problem 4.26 for $u_3^{(1)}$. We make the boundary conditions homogeneous. This introduces an additional forcing term g_{BC}. We can estimate all norms of g_{BC} in terms of the inhomogeneity of the boundary conditions. Writing the boundary conditions in the original variables we have the problem

$$\begin{cases} u_{3x}^{(1)} = B^{-1}Au_3^{(1)} + g^{(1)} \\ \left[R(l) + \mathcal{O}(s)\right] u_3^{(1)}(l) = 0 \\ \left[R(-l) + \mathcal{O}(s)\right] u_3^{(1)}(-l) = 0 \end{cases} \tag{4.31}$$

where R was introduced in section 3.3 and

$$g^{(1)} = B^{-1}v_3^{(1)} + sg_M + g_{BC}.$$

We have the inequality

$$\|g^{(1)}\|_{L^2[-l,l]} \leq sK_2(l)\left(\|h_M\|_{L^2} + \|g_M\|_{L^2}\right).$$

We treated of problem of the type 4.31 in section 3.3. According to lemma 10 we can construct a solution satisfying

$$\|u_3^{(1)}\|_{L^\infty[-l,l]} + \|u_3^{(1)}\|_{H^1[-l,l]} \leq sK_3(l)\left(\|h_M\|_{L^2} + \|g_M\|_{L^2}\right). \tag{4.32}$$

This completes the construction and estimates for the first iteration. The treatment of the i:th iterate is similar. First we determine $v_3^{(i)}$ and $\alpha^{(i)}$. The boundary conditions give

$$\begin{cases} v_3^{(i)}(l) = \left[S_{1R}^{II} + \mathcal{O}(s)\right] \bar{v}_3^{(i)II}(l) + b^{(i-1)}(l) \\ v_3^{(i)}(-l) = \left[S_{1L}^{I} + \mathcal{O}(s)\right] \bar{v}_3^{(i)I}(-l) + b^{(i-1)}(-l) \end{cases}$$

where

$$|b^{(i-1)}(l)| + |b^{(i-1)}(-l)| \leq sK_4 \left(\|u_3^{(i-1)}\|_{L^\infty[-l,l]} + \|v_3^{(i-1)}\|_{L^\infty[-l,l]} \right).$$

The integration of the equation for $v_3^{(i)}$ is done as before. The linear system of equations for $\bar{v}_3^{(i)II}(l)$, $\bar{v}_3^{(i)I}(-l)$ and $\alpha^{(i)}$ becomes

$$\begin{aligned} &\left(S_{2R}^{II} + \mathcal{O}(s)\,;\, -S_{1L}^{I} + \mathcal{O}(s)\,;\, U_R - U_L + \mathcal{O}(e^{-\beta l})\right) \begin{pmatrix} \bar{v}_3^{(i)II}(l) \\ \bar{v}_3^{(i)I}(-l) \\ s\alpha^{(i)} \end{pmatrix} \\ &\qquad = s\int_{-l}^{l} u_3^{(i-1)} dx + b^{(i-1)}(l) + b^{(i-1)}(-l) \end{aligned}$$

The system is solved and we get estimates

$$\begin{aligned} &\|v_3^{(i)}\|_{L^\infty[-l,l]} + \|v_3^{(i)}\|_{H^1[-l,l]} + |s\alpha^{(i)}| \\ &\quad \leq sK_5(l) \left(\|u_3^{(i-1)}\|_{L^2[-l,l]} + \|u_3^{(i-1)}\|_{L^\infty[-l,l]} + \|v_3^{(i-1)}\|_{L^\infty[-l,l]} \right). \end{aligned} \tag{4.33}$$

Finally we determine $u_3^{(i)}$. We make the boundary conditions homogeneous and write them in the original variables. This gives the problem

$$\begin{cases} u_{3x}^{(i)} = B^{-1}Au_3^{(i)} + g^{(i)} \\ \left[R(l) + \mathcal{O}(s)\right] u_3^{(i)}(l) = 0 \\ \left[R(-l) + \mathcal{O}(s)\right] u_3^{(i)}(-l) = 0 \end{cases}$$

where

$$g^{(i)} = B^{-1}v_3^{(i)} + g_{BC}$$

and we can estimate g_{BC} according to

$$\|g_{BC}\|_{L^2[-l,l]} \leq sK_6(l) \left(\|u_3^{(i-1)}\|_{L^2[-l,l]} + \|u_3^{(i-1)}\|_{L^\infty[-l,l]} + \|v_3^{(i-1)}\|_{L^\infty[-l,l]} \right).$$

Referring to section 3.3 and lemma 10 we can construct a solution satisfying

$$\begin{aligned} &\|v_3^{(i)}\|_{L^\infty[-l,l]} + \|v_3^{(i)}\|_{H^1[-l,l]} \\ &\quad \leq sK_7(l) \left[\|u_3^{(i-1)}\|_{L^2[-l,l]} + \|v_3^{(i-1)}\|_{L^2[-l,l]} \right] \end{aligned} \tag{4.34}$$

$$+sK_7(l)\left[\|u_3^{(i-1)}\|_{L^\infty[-l,l]} + \|v_3^{(i-1)}\|_{L^\infty[-l,l]}\right].$$

The construction is complete. We see that for $|s|$ sufficiently small the iteration defines a contraction. We sum the iterates

$$u_3 = \sum_{i=1}^{\infty} u_3^{(i)}, \qquad v_3 = \sum_{i=1}^{\infty} v_3^{(i)}, \qquad \alpha = \sum_{i=1}^{\infty} \alpha^{(i)}$$

and get a solution of equation 4.25. Summing the estimates 4.30, 4.32, 4.33, 4.34 for the iterates we obtain estimates for the solution:

$$\begin{aligned} \|u_3\|_{H^1[-l,l]} &\leq sK(l)\left(\|h_M\|_{L^2} + \|g_M\|_{L^2}\right) \\ |\alpha| &\leq K(l)\left(\|h_M\|_{L^2} + \|g_M\|_{L^2}\right). \end{aligned} \tag{4.35}$$

We also have an estimate for v_3 but that is not needed for later purposes.

4.2.4 Summary of the splittings and completion of the estimate

Before we summarize the splittings we have done on u we extend the function u_3 to the intervals $(-\infty, -l]$ and $[l, \infty)$. This is done with the homogeneous equation 4.25. When we do the extension in this way the result is that u_3, α solves the equation

$$\begin{pmatrix} u_3 \\ v_3 \end{pmatrix}_x = \begin{pmatrix} B^{-1}A & B^{-1} \\ sI & 0 \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} + s \begin{pmatrix} g_M \\ h_M - \alpha\tilde{\varphi}_0 \end{pmatrix}, \qquad x \in \mathbb{R}$$

where

$$\tilde{\varphi}_0 = \begin{cases} \varphi_0, & |x| \leq l_1 \\ 0 & |x| \geq l_1 \end{cases}$$

For the extension we use lemma 19, this gives estimates which, taken together with 4.35, are

$$\|u_3\|_{H^1} \leq K(l)\left(\|h_M\|_{L^2} + \|g_M\|_{L^2}\right). \tag{4.36}$$

The splittings we have introduced in this section are related by

$$u = u_1 + \varphi_R u_{2R} + \varphi_L u_{2L} + u_3 + \alpha\varphi_0 + u_4.$$

Here the function u_4 satisfies the equation

$$su_4 + (Au_4)_x = Bu_{4xx} + sh_4$$

where $h_4 = \alpha(\varphi_0 - \tilde{\varphi}_0)$. This means that the function u_4 satisfies the same type of problem as u_2, see equation 4.15. The difference being that the norms of the forcing have been reduced by a factor $\mathcal{O}(e^{-\beta l})$. This leads us to define an iteration to determine u. The first iterate $u^{(1)}$ is given by

$$u^{(1)} = u_1 + \varphi_R u_{2R} + \varphi_L u_{2L} + u_3 + \alpha\varphi_0.$$

Then we repeat the process and split u_4 according to

$$u_4 = \varphi_R u_{2R}^{(2)} + \varphi_L u_{2L}^{(2)} + u_3^{(2)} + \alpha^{(2)}\varphi_0 + u_4^{(2)}$$

and define the second iterate by

$$u^{(2)} = u_4 - u_4^{(2)}.$$

The process is continued, we have

$$u_4^{(i-1)} = \varphi_R u_{2R}^{(i)} + \varphi_L u_{2L}^{(i)} + u_3^{(i)} + \alpha^{(i)}\varphi_0 + +u_4^{(i)}.$$

Because of the reduction of the forcing in each step we have, for sufficiently large l

$$\lim_{i\to\infty} \|u_4^{(i)}\|_{L^\infty} = 0.$$

In conclusion, we have constructed a solution of the resolvent equation 4.1,

$$u = \sum_{i=1}^{\infty} u^{(i)}$$

for Re $s \geq 0$, $0 < |s| \leq \delta$. Summing the estimates of the iterates using lemma 15 and lemma 16 and equation 4.35 and 4.36 we obtain

$$\|u\|_{H^1} \leq K\left(\|h\|_{L^1} + \|h\|_{L^2}\right)$$

and the proof of lemma 14 is complete.

Chapter 5

Numerical investigation of the eigenvalue condition

In this chapter assumption 5 is investigated numerically. This is done in two cases. First we consider the Burgers equation, here the stability of shocks can be proven analytically. The reason we include this case is to introduce the numerical methods in a simple setting. In the second section we investigate the Navier-Stokes equations with the physical viscosity replaced by a scaled identity matrix. The computations are carried out in Matlab.

The object is to investigate whether there are any eigenvalues in the right half plane. For the computation we need to replace the eigenvalue problem for $x \in \mathbb{R}$ with a problem on a bounded domain. This truncation changes the spectrum, for example the problem on a bounded domain has no continuous spectrum. For the purpose of our investigation this is however irrelevant because the continuous spectrum is situated in the left half plane. Also, if we have an eigenvalue in the right half plane, the corresponding eigenfunction decays exponentially. This follows from to the properties of κ_i derived in chapter 3. This means that we can choose as boundary conditions on the bounded interval that the function is zero.

In summary, if the eigenvalue problem have an eigenvalue in the right half plane, the truncated problem will have an eigenvalue converging to it when the truncation interval is made larger. This argument is made rigorous for a similar problem in [4]

5.1 Burgers' equation

Here we consider the viscous Burgers' equation

$$u_t + \left(u^2\right)_x = u_{xx}. \tag{5.1}$$

The numerical investigation consists of two parts, first to find a shock profile and then to calculate the eigenvalues with largest real part of the problem linearized at the profile. We use the same discretization of space for both parts. We must of course compute on a finite x-interval, denoted $[0, L]$. We will study how this truncation affects the results. The x-axis is discretized according to

$$\begin{aligned} x_i &= i\Delta x \\ \Delta x &= \frac{L}{N+1}. \end{aligned}$$

For equation 5.1 we even have an analytical expression for the shock profiles, they are tangenshyperbolicus curves connecting the left and right values. We supplement equation 5.1 with the boundary conditions $u(0) = -u(L) = 1$. According to the Rankine-Hugoniot relation this will give a stationary shock. For initial data we choose a straight line $u(x, 0) = 1 - 2x/L$ connecting $u(0) = 1$ and $u(L) = -1$.

5.1.1 Calculation of the shock profile

We use a finite difference approximation. We discretize the space derivatives in 5.1 with second order accuracy. This leads to the semi-discrete problem

$$\begin{aligned} \frac{du_1}{dt} &= -\frac{u_2^2 - 1}{2\Delta x} + \frac{u_2 - 2u_1 + 1}{\Delta x^2} \\ \frac{du_i}{dt} &= -\frac{u_{i+1}^2 - u_{i-1}^2}{2\Delta x} + \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}, \qquad i = 2, \ldots, N-1 \\ \frac{du_N}{dt} &= -\frac{1 - u_{N-1}^2}{2\Delta x} + \frac{-1 - 2u_N + u_{N-1}}{\Delta x^2}. \end{aligned} \tag{5.2}$$

Here we have also taken the boundary conditions into account. We rewrite 5.2 in vector notation

$$\frac{dU}{dt} = F(U)$$

where $U = (u_1, \ldots, u_N)^T$. The time-stepping is done with the backward Euler method

$$\frac{U^{n+1} - U^n}{\Delta t} = F(U^{n+1})$$

where n is the time index. In each time step we must solve a system of non-linear equations, this is done with Newton's method. The Jacobian J of F is then needed and we calculate the expression for it analytically. In each step of Newton's method we must invert a tridiagonal matrix so the complexity is $\mathcal{O}(N)$.

There are no problems in the convergence of Newton's method. The time-stepping also works well so it is possible to use large values of Δt, the time-step. In the computations presented here we have $\Delta t \sim 10^6$. We then need about 3-7 time-steps to bring down $F(U^n)$, the residual, to machine precision. Calculations have been done with $L \in [10, 100]$ and $N \in [100, 10000]$. In figure 5.1 we see the resulting shock profile and residual for a typical computation.

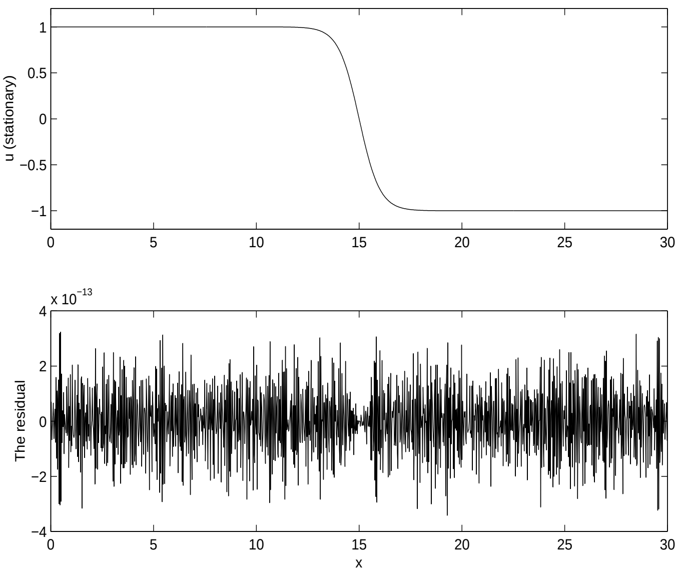

Figure 5.1. The computed shock profile and residual for L=30 and N=1200.

5.1.2 Computing the eigenvalues

By the procedure described above we determine a stationary solution. We will now calculate the eigenvalues of the linearized problem. In the discrete problem this means calculating the eigenvalues of the $N \times N$ matrix $J(U)$ where U is the stationary solution. The eigenvalues of the continuous problem are real because it is self-adjoint.

We use the Matlab function "eigs" to calculate the eigenvalues. We give parameters so that a shift and invert iteration is performed with the implicitly restarted Arnoldi factorization, see [10]. The shift is chosen to -0.4 which is between the two largest eigenvalues. Convergence for the ten largest eigenvalues is obtained in typically 1-3 restarts of the Arnoldi method. The calculation involves one inversion of the matrix J. To determine the eigenvalues requires less computations than the time-stepping and has complexity $\mathcal{O}(N)$, unless convergence problems arise. This has not been the case in any of the test runs we have made.

In the table below we study the convergence of the eigenvalues when the grid is refined and the truncation interval is made larger. I e we study the limits $\Delta x \to 0$ and $L \to \infty$. Results for the second and third largest eigenvalues are presented. The largest eigenvalue, which we know is zero, converges quickly.

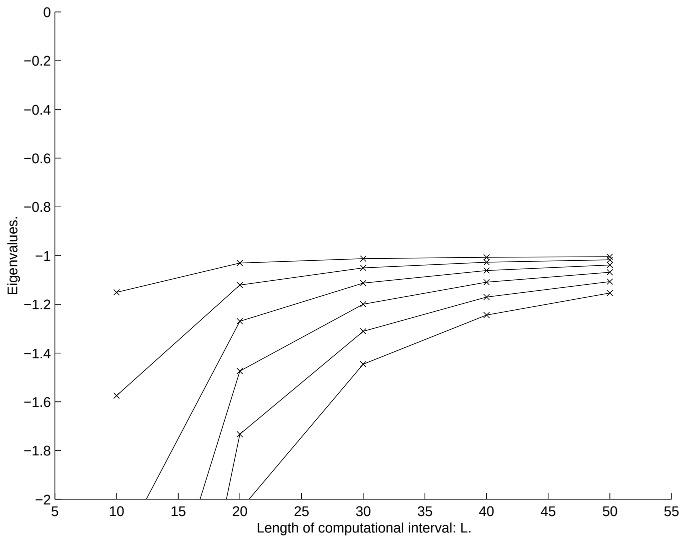

Figure 5.2. Asymptotic behaviour for large L of the six largest eigenvalues.

	2nd eigenvalue			
	L=10	L=20	L=30	L=40
N/L=10	-1.1511	-1.0320	-1.0145	-1.0089
N/L=20	-1.1510	-1.0308	-1.0131	-1.0074
N/L=40	-1.1509	-1.0305	-1.1027	-1.0070
N/L=80	-1.1509	-1.0304	-1.0126	-1.0069
	3rd eigenvalue			
	L=10	L=20	L=30	L=40
N/L=10	-1.5743	-1.1227	-1.0525	-1.0297
N/L=20	-1.5747	-1.1213	-1.0508	-1.0279
N/L=40	-1.5748	-1.1210	-1.0504	-1.0275
N/L=80	-1.5748	-1.1209	-1.0503	-1.0274

In figure 5.2 we illustrate the behaviour of the six largest eigenvalues as $L \to \infty$. The figure clearly indicates that the original problem have no eigenvalues with real part larger than zero.

5.2 The modified Navier-Stokes equations

Here we consider the Navier-Stokes equations with artificial viscosity replacing the physical viscosity according to

$$u_t + f_x(u) = \nu u_{xx} \tag{5.3}$$

where $u = (\rho, m, e)^T$ and ρ is the density, m is the momentum and e is the total energy. The function f is given by

$$f = \begin{pmatrix} m \\ \frac{m^2}{\rho} + p \\ (e + p)\frac{m}{\rho}) \end{pmatrix}$$

where $\gamma = 1.4$ is the gas constant and p is the pressure which is related to the other variables by

$$p = (\gamma - 1)(e - \frac{m^2}{2\rho}).$$

We employ the same numerical methods as in the Burgers case.

5.2.1 Calculation of the shock profile.

The x-interval is truncated to $[0, L]$ which then is discretized into N points. Because 5.3 is a system of equations the discrete problem has $3N$ unknowns. Writing the semi-discrete problem on vector form

$$\frac{dU}{dt} = F(U)$$

where the unknowns are sorted so that $U = (\rho_1, m_1, e_1, \rho_2, m_2, \ldots, \rho_N, m_N, e_N)^T$. To find boundary conditions we must find left and right states which are connected by a Lax shock. We choose a supersonic left state $u_L = (\rho_L, m_L, e_L)^T$ and set the shock speed to zero in the Rankine-Hugoniot relations to get

$$f(u_L) = f(u_R).$$

These equations are solved for u_R. Then we must check that the right state actually has one characteristic entering the shock. This procedure poses no problem and the computations are done with the values

$$\begin{cases} \rho_L = 0.5800 \\ m_L = 0.9608 \\ e_L = 2.2458 \end{cases} \quad \begin{cases} \rho_R = 0.9800 \\ m_R = m_L \\ e_R = 3.5450. \end{cases}$$

For numerical boundary conditions we now set $u_0 = (\rho_L, m_L, e_L)^T$ and $u_{N+1} = (\rho_R, m_R, e_R)^T$. The computation we present results for in the next section uses

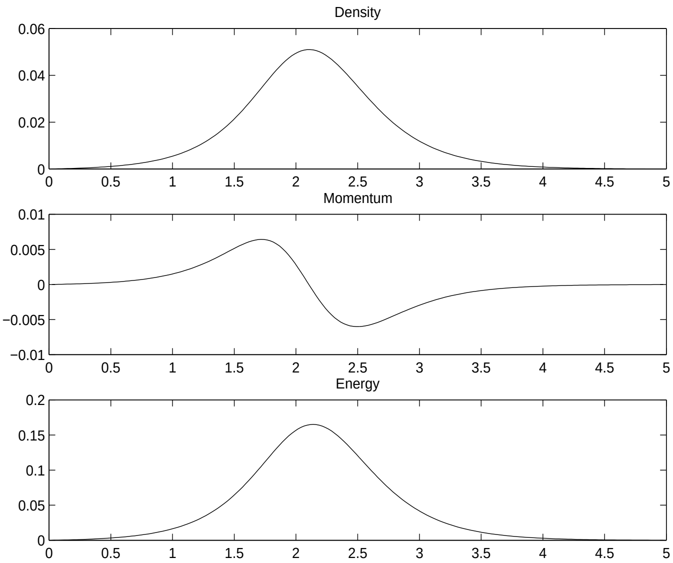

Figure 5.3. The eigenfunction corresponding to the zero eigenvalue.

the parameter values $\nu = 0.1$, $L = 5$ and $N = 200$. The convergence is considerably slower than in the Burgers case. Of course, the convergence of the time stepping and Newton's method are related, large time steps may cause problems for Newton's method. Choosing the time steps carefully it is possible to bring down the residual to machine precision. This requires about 50 time steps, each of which needs 10-20 Newton iterations.

5.2.2 Computing the eigenvalues

We calculate the eigenvalues of $J(U)$ where J is the Jacobian of F and U is the computed shock profile. Because of the ordering of the unknowns J has band width six. I e $a_{ij} = 0$ if $|i - j| \geq 6$ where a_{ij} are the components of J. This is important because the inverse iteration requires inversion of the matrix. In figure 5.3 and 5.4 we present the results of the test run described above. Figure 5.3 shows the eigenfunction of the zero eigenvalue. In figure 5.4 we have plotted the eigenvalues of largest real part.

The inverse iteration with the Arnoldi method works well in the Navier-Stokes case too. For these algorithms and this problem the limiting factor for computations with larger N and L is clearly the calculation of the shock profile.

The theory for the discretization [5], says roughly that the eigenvalues corresponding to eigenfunctions which are well represented on the grid are computed with $\mathcal{O}(dx^2)$ accuracy. We also argue that all eigenvalues with small absolute

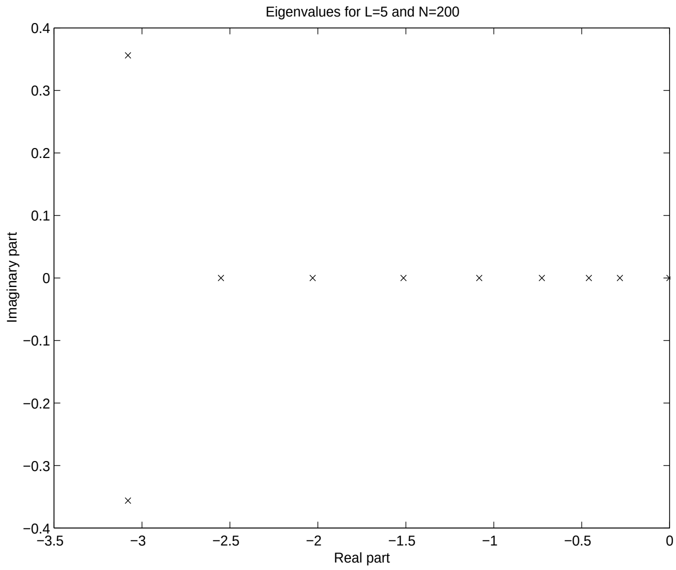

Figure 5.4. Eigenvalues.

value corresponds to eigenfunctions which vary slowly and thus they can be computed with a reasonable size of N. We recall that in the right half plane the parabolicity of the equation implies that the eigenvalues can only lie in a bounded region.

Appendix A

The linearization procedure

Here we derive the results presented in section 2.2. We have the equation

$$v_t + f(v)_x = Bv_{xx}, \qquad (x,t) \in \mathbb{R} \times [0, \infty) \tag{A.1}$$

and make the substitution

$$v(x,t) = U(x) + \epsilon v_{0x}(x)e^{-t} + \epsilon u(x,t,\epsilon).$$

Here all the notation was introduced in chapter 2. A priori we assume that u is a smooth uniformly bounded function, this assumption will be verified by the stability proof. We Taylor expand f to get

$$f(v) = f(U + \epsilon v_{0x}e^{-t}) + \epsilon f'(U + \epsilon v_{0x}e^{-t})u + \epsilon^2 g(x,t,u,\epsilon) \tag{A.2}$$

where the rest term g is bounded according to

$$|g(x,t,u,\epsilon)| < K|u|^2.$$

Next we Taylor expand the first term in the right hand side of A.2

$$f(U + \epsilon v_{0x}e^{-t}) = f(U) + \epsilon h_1(x,t,\epsilon). \tag{A.3}$$

The notational convention we use is to label non-linear terms by g_i and terms not depending on u by h_i, we use lower-case letters for vectors and capital letters for matrices. Finally we Taylor expand the second term in the right hand side of A.2

$$\epsilon f'(U + \epsilon v_{0x}e^{-t})u = \epsilon f'(U)u + \epsilon^2 H(x,t,\epsilon)u. \tag{A.4}$$

We now collect equation A.2, A.3 and A.4 and insert in equation A.1

$$\begin{aligned}&\epsilon u_t - \epsilon v_{0x}e^{-t} + f_x(U) + \epsilon h_{1x} + \epsilon[f'(U)u]_x + \epsilon^2(Hu)_x + \epsilon^2 g_x \\ &= BU_{xx} + \epsilon Bv_{0xxx}e^{-t} + \epsilon Bu_{xx}.\end{aligned} \tag{A.5}$$

Since U is a solution of A.1 we have $f_x(U) = BU_{xx}$. We divide equation A.5 with ϵ, rearrange the terms and introduce new notation to arrive at

$$u_t + [A(x)u]_x = Bu_{xx} + h_x(x,t,\epsilon) - \epsilon[g(x,t,u,\epsilon) + H(x,t,\epsilon)u]_x$$

where

$$\begin{aligned} A &= f'(U) \\ h &= -h_1 + (v_0 + Bv_{0xx})e^{-t}. \end{aligned}$$

We also need estimates for h, H and g. We have the expression

$$h_1(x,t,\epsilon) = \frac{1}{\epsilon}\left(f(U) - f(U + \epsilon v_{0x}e^{-t}\right)$$

from which, by Taylors formula, we get

$$|h_1(x,t,\epsilon)| \leq K_1|v_{0x}e^{-t}| \tag{A.6}$$

where K_1 is the largest absolute value of f' where the argument takes values. Integrating A.6 with respect to x we have

$$\|h_1(\cdot,t,\epsilon)\|_{L^1} \leq K_1 e^{-t}\|v_{0x}\|_{L^1}.$$

Integrating this with respect to t gives

$$\int_0^\infty \|h_1(\cdot,t,\epsilon)\|_{L^1} dt \leq K_1\|v_{0x}\|_{L^1}.$$

According to assumption 2, v_{0x} and all its derivatives are in L^1 and L^2 so we have

$$\int_0^\infty \|h_1(\cdot,t,\epsilon)\|_{L^1} dt \leq K_2.$$

The inequalities for h, H and g given in section 2.2 are derived in a similar manner.

Appendix B

Elementary estimates for ODE:s on a half-line

In this appendix we derive results on ODE's defined on an interval $[l, \infty)$. The techniques developed here are applied in chapter 3 and 4 where we replace $\mathbb{R}$ by a bounded interval $[-l, l]$.

We start with a scalar constant coefficient ODE. Then we study a system of ODE:s with constant coefficients. After that we have a coefficient matrix which is nearly constant. In the last lemma we study the case when the coefficient matrix depends analytically on a parameter.

Lemma 17 *Consider the ODE*

$$\frac{du}{dx} = \lambda u + f, \qquad x \in [0, \infty) \tag{B.1}$$

where $f \in C^1 \cap L^1$ and $\lambda \in \mathbb{C}$. We state two results, for real part of λ negative and positive respectively. If $\mathrm{Re}\,\lambda < 0$ and we have the initial condition $u(0) = u_0$, then the solution satisfies the estimates

$$\begin{aligned} \|u\|_{L^1} &\leq \frac{1}{|\mathrm{Re}\,\lambda|}\left(|u_0| + \|f\|_{L^1}\right) \\ \|u\|_{L^2} &\leq \frac{1}{|\mathrm{Re}\,\lambda|^{1/2}}\left(|u_0| + \|f\|_{L^1}\right) \\ \|u\|_{L^\infty} &\leq |u_0| + \|f\|_{L^1}. \end{aligned}$$

If $\mathrm{Re}\,\lambda > 0$ then the solution is uniquely determined by equation B.1 and the additional condition that the solution is a bounded function. The solution satisfies the estimates

$$\|u\|_{L^1} \leq \frac{1}{|\mathrm{Re}\,\lambda|}\|f\|_{L^1}$$

$$\begin{aligned} \|u\|_{L^2} &\leq \frac{1}{|\text{Re }\lambda|^{1/2}}\|f\|_{L^1} \\ \|u\|_{L^\infty} &\leq \|f\|_{L^1}. \end{aligned} \tag{B.2}$$

Proof: The solution is in both cases given by the expression

$$u(x) = u(0)e^{\lambda x} + \int_0^x e^{\lambda(x-\xi)} f(\xi)d\xi. \tag{B.3}$$

We first consider the case Re $\lambda < 0$. The L^1-estimate follows from

$$\begin{aligned} \|u\|_{L^1} &\leq |u_0| \int_0^\infty e^{\text{Re }\lambda x} dx + \int_0^\infty \left| \int_0^x e^{\lambda(x-\xi)} f(\xi)d\xi \right| dx \\ &\leq \frac{1}{|\text{Re }\lambda|}|u_0| + \int_0^\infty |f(\xi)| \int_\xi^\infty e^{\text{Re }\lambda(x-\xi)} dx d\xi \\ &\leq \frac{1}{|\text{Re }\lambda|}\left(|u_0| + \|f\|_{L^1}\right) \end{aligned}$$

To estimate the L^2-norm we first use the triangle inequality. The norm of the exponential function is trivial, to get the estimate of the second term we use the following series of inequalities

$$\begin{aligned} &\int_0^\infty \left(\int_0^x e^{\text{Re }\lambda(x-\xi)} |f(\xi)|d\xi \right)^2 dx \\ &\quad \leq \|f\|_{L^1} \int_0^\infty \int_0^x e^{\text{Re }\lambda(x-\xi)} |f(\xi)| d\xi dx \leq \frac{1}{|\text{Re }\lambda|}\|f\|_{L^1}^2 \end{aligned}$$

where we change the order of integration to get the last inequality. The L^∞-inequality follows immediately by taking the absolute value under the integration and using Re $\lambda < 0$.

In the case Re $\lambda > 0$ we must have

$$u(0) = -\int_0^\infty e^{-\lambda\xi} f(\xi)d\xi \tag{B.4}$$

all other initial data give a solution that grows for $x \to \infty$. Inserting B.4 into B.3 we get the following expression for the solution

$$u(x) = -\int_x^\infty e^{\lambda(x-\xi)} f(\xi)d\xi.$$

Now the same techniques as in the case Re $\lambda < 0$ can be applied to derive the estimates B.2.

Lemma 18 *Consider the system of ODE's*

$$\frac{du}{dx} = Au + f, \;\; x \in [0.\infty) \tag{B.5}$$

where f is a vector of n functions in $C^1 \cap L^1$ and A is a $n \times n$ complex matrix. We state two results, one when all eigenvalues of A have negative real part and one when all have positive real part.

If Re $\lambda_i < 0$ *for all eigenvalues* λ_i *of* A *and we have initial data* $u(0) = u_0$, *then we have a constant* K *only depending on* A *such that*

$$\begin{aligned} \|u\|_{L^1} &\le \frac{K}{|\mathrm{Re}\ \lambda|}(|u_0| + \|f\|_{L^1}) \\ \|u\|_{L^2} &\le \frac{K}{|\mathrm{Re}\ \lambda|^{1/2}}(|u_0| + \|f\|_{L^1}) \\ \|u\|_{L^\infty} &\le K(|u_0| + \|f\|_{L^1}). \end{aligned} \tag{B.6}$$

where $|\mathrm{Re}\ \lambda| = \min_i |\mathrm{Re}\ \lambda_i|$

If Re $\lambda_i > 0$ *for all* i *then the solution is uniquely determined by equation B.5 and the additional condition that the solution is bounded. The solution satisfies the estimates*

$$\begin{aligned} \|u\|_{L^1} &\le \frac{K}{|\mathrm{Re}\ \lambda|}\|f\|_{L^1} \\ \|u\|_{L^2} &\le \frac{K}{|\mathrm{Re}\ \lambda|^{1/2}}\|f\|_{L^1} \\ \|u\|_{L^\infty} &\le K\|f\|_{L^1}. \end{aligned}$$

Proof: We will use the Jordan canonical form J of the matrix A. Let S be a transformation matrix such that

$$S^{-1}AS = J.$$

The substitution $u = Sv$ then decouples B.5 into equations of the form

$$\frac{dw}{dx} = J_r w + g \tag{B.7}$$

where J_r is a Jordan block of size r corresponding to the eigenvalue λ. The solution operator of equation B.7 is

$$S_r(x,\xi) = e^{\lambda(x-\xi)} \begin{pmatrix} 1 & x-\xi & \dots & \frac{(x-\xi)^{r-1}}{(r-1)!} \\ 0 & 1 & \dots & \frac{(x-\xi)^{r-2}}{(r-2)!} \\ . & . & \dots & . \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

We can now write down the solution of B.7. In the case Re $\lambda < 0$ we supplement B.7 with initial conditions $w(0) = w_0$ and have

$$w(x) = S_r(x,0)w_0 + \int_0^x S_r(x,\xi)g(\xi)d\xi$$

Now estimates of the type B.6 can be derived for w. In the scalar case we had the simple relation

$$|e^{\lambda(x-\xi)}| = e^{\mathrm{Re}\ \lambda(x-\xi)}.$$

This now has to be replaced by

$$|S_r(x,\xi)| < K_1 e^{\frac{\mathrm{Re}\,\lambda}{2}(x-\xi)} \tag{B.8}$$

where K_1 only depend on A. Using B.8 the derivation is now very similar to the scalar case and we obtain estimates of w in terms of the L^1-norm of g. Returning to the original variables u and f we get B.6 and see that K depends on K_1, $|S|$, $|S^{-1}|$ and the number of Jordan blocks. In short K depends on A.

Using the remarks above the the derivation in the case $\mathrm{Re}\,\lambda_i > 0$ is also similar to that for the scalar equation. We do not give this part of the proof.

Lemma 19 *Consider the system of ODE's*

$$\frac{du}{dx} = \left[\begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix} + e^{-\beta x} B(x)\right] u + f, \qquad x \in [l,\infty) \tag{B.9}$$

where $l > 0$ and the coefficients and the forcing satisfies the following assumptions

- *The matrices A^+ and A^- are square, the eigenvalues of A^+ have positive real part and the eigenvalues of A^- have negative real part.*
- *The matrix function B is uniformly bounded, $\beta > 0$ and $f \in C^1 \cap L^1$.*

Equation B.9 is studied for $x \in [l,\infty)$ and we have the condition that the solution is a bounded function and satisfies the initial condition

$$u^{II}(l) = u_0^{II} \tag{B.10}$$

where u^{II} denotes the r last components of u and r is the size of A^-. Under the above assumptions there exists an l_0 such that for $l \geq l_0$ the solution is unique and we have a constant K such that

$$\begin{aligned} \|u\|_{L^1} &\leq \frac{K}{|\mathrm{Re}\,\lambda|}(|u_0| + \|f\|_{L^1}) \\ \|u\|_{L^2} &\leq \frac{K}{|\mathrm{Re}\,\lambda|^{1/2}}(|u_0| + \|f\|_{L^1}) \\ \|u\|_{L^\infty} &\leq K(|u_0| + \|f\|_{L^1}). \end{aligned}$$

Here $|\mathrm{Re}\,\lambda|$ is defined by

$$|\mathrm{Re}\,\lambda| = \min\left(\min_{1\leq i\leq n-r} |\mathrm{Re}\,\lambda_i^+|, \min_{n-r+1\leq i\leq n} |\mathrm{Re}\,\lambda_i^-|\right)$$

where λ_i^+ are the eigenvalues of A^+ and λ_i^- the eigenvalues of A^-.

Proof: We construct the solution by an iteration where the first iterate solves the equation

$$\frac{du^{(1)}}{dx} = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} u^{(1)} + f, \qquad x \in [l, \infty) \tag{B.11}$$

with boundary conditions

$$u^{II(1)}(0) = u_0^{II} \tag{B.12}$$

and the additional condition that the solution is bounded. The i:th iterate solves the equation

$$\frac{du^{(i)}}{dx} = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} u^{(i)} + e^{-\beta x} B(x) u^{(i-1)}, \qquad x \in [l, \infty) \tag{B.13}$$

with boundary conditions

$$u^{II(i)}(0) = 0 \tag{B.14}$$

and the requirement that the solution is bounded. Using lemma 18 on the problem B.11, B.12 we have a constant K_1 such that

$$\|u^{(1)}\|_{L^\infty} \leq K_1 \left(|u_0^{II}| + \|f\|_{L^1} \right)$$

where the norms of course are taken over the interval $[l, \infty)$. Applying lemma 18 two the problem B.13, B.14 we have

$$\|u^{(i)}\|_{L^\infty} \leq K_1 \|e^{-\beta x} B(x) u^{(i-1)}\|_{L^1} \leq K_1 \|e^{-\beta x} B(x)\|_{L^1} \|u^{(i-1)}\|_{L^\infty} \tag{B.15}$$

We now choose l_0 so that

$$\|e^{-\beta x} B(x)\|_{L^1[l_0,\infty)} = \frac{1}{2K_1}$$

Using this in B.15 we see that the iteration is a L^∞- contraction. The function

$$u = \sum_{i-1}^{\infty} u^{(i)}$$

solves the problem B.9, B.10 and summing the estimates we get

$$\|u\|_{L^\infty} \leq 2K_1 \left(|u_0^{II}| + \|f\|_{L^1} \right) \sum_{i=1}^{\infty} \frac{1}{2^i} \leq 4K_1 \left(|u_0^{II}| + \|f\|_{L^1} \right) \tag{B.16}$$

To obtain the L^1-estimate we study the problem B.9, B.10 and consider $e^{-\beta x} B(x)$ part of the forcing when applying lemma 18.

$$\|u\|_{L^1} \leq \frac{2K_1}{|\text{Re } \lambda|} \left(|u_0^{II}| + \|e^{-\beta x} B(x) u + f\|_{L^1} \right)$$

$$\leq \frac{2K_1}{|\text{Re }\lambda|}\left(|u_0^{II}| + \|f\|_{L^1} + \|e^{-\beta x}B(x)\|_{L^1}\|u\|_{L^\infty}\right)$$
$$\leq \frac{4K_1}{|\text{Re }\lambda|}\left(|u_0^{II}| + \|f\|_{L^1}\right)$$

Here the last inequality is obtained by using B.16 and $l \geq l_0$. The L^2-estimate is derived in the same manner as the L^1-estimate. We do not give this part of the proof.

Lemma 20 *Consider the system of ODE's B.9 with forcing $f = 0$ and boundary conditions B.10. According to lemma 19 , for $l \geq l_0$, we have a unique solution. The problem then defines a linear mapping*

$$u^I(l) = Q(l)u^{II}(l)$$

The result of this lemma is that there exist a positive constant K such that

$$|Q(l)| \leq Ke^{-\beta l}$$

for $l \geq l_0$.

Proof: We insert the ansatz

$$u = \begin{pmatrix} 0 \\ e^{A^- x}u_0^{II} \end{pmatrix} + v$$

in equation B.9. The forcing is zero and we obtain the equation

$$\frac{dv}{dx} = \left[\begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix} + e^{-\beta x}B(x)\right] v + e^{-\beta x}B(x)\begin{pmatrix} 0 \\ e^{A^- x}u_0^{II} \end{pmatrix}$$

and the boundary conditions

$$v_0^{II}(l) = 0$$

for the new unknown v. Using lemma 19 we obtain the estimate

$$\|v\|_{L^\infty[l,\infty)} \leq K_1\|e^{-\beta x}B(x)\begin{pmatrix} 0 \\ e^{A^- x}u_0^{II} \end{pmatrix}\|_{L^1[l,\infty)} \leq Ke^{-\beta l}|u_0^{II}|.$$

We also have $|u^I(l)| \leq \|v\|_{L^\infty[l,\infty)}$, which completes the proof.

Lemma 21 *Consider the system of ODE's*

$$\frac{du}{dx} = \left[\begin{pmatrix} A^+(s) & 0 \\ 0 & A^-(s) \end{pmatrix} + e^{-\beta x}B(x,s)\right] u + f, \qquad x \in [l,\infty) \tag{B.17}$$

with boundary conditions

$$u^{II}(l) = u_0^{II}. \tag{B.18}$$

Here A^+, A^- and B are analytic at $s = 0$ for fixed x. Also, $A^+(0)$, $A^-(0)$ and $B(x,0)$ satisfy the requirements in lemma 19. The linear mapping

$$u^I(l,s) = Q(l,s)u_0^{II}$$

defined by B.17, B.18 then also is analytic at $s = 0$.

Proof: First consider the problem when A^+ and A^- do not depend on s. The linear mapping defined by this problem is denoted $Q_0(l,s)$. We differentiate the ODE and the boundary conditions with respect to s and obtain

$$\begin{aligned} \left(\frac{\partial u}{\partial s}\right)' &= \left[\begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix} + e^{-\beta x}B(x,s)\right]\frac{\partial u}{\partial s} + e^{-\beta x}\frac{\partial B}{\partial s}u \\ \frac{\partial u}{\partial s}^{II}(l) &= 0 \end{aligned}$$

where prime denotes differentiation with respect to x. We now apply lemma 19 to the problem for u and to the problem for $\frac{\partial u}{\partial s}$. This leads to a uniform bound for $\frac{\partial u}{\partial s}(l,s)$ in a neighborhood of $s=0$. This means that the linear mapping $Q_0(l,s)$ is analytic at $s=0$ for fixed l. Now we return to the case when A^+ and A^- depend on s. We have

$$\begin{aligned} A^+ &= A_0^+ + sA_1^+(s) \\ A^- &= A_0^- + sA_1^-(s). \end{aligned}$$

We introduce a new variable, $\tilde{u}$, by

$$u = e^{sA_1x}\tilde{u}$$

where

$$A_1 = \begin{pmatrix} A_1^+ & 0 \\ 0 & A_1^- \end{pmatrix}.$$

We insert this expression in $B.17$. This gives

$$\tilde{u}' = \left[\begin{pmatrix} A_0^+ & 0 \\ 0 & A_0^- \end{pmatrix} + e^{-\beta x}B(x,s)\right]\tilde{u}.$$

and we have reduced the problem to a problem of the type treated first. The x-independent part of the coefficient matrix do not depend on s. This means that $\tilde{u}^I(l,s)$ and therefore $u^I(l,s)$ and $Q(l,s)$ are analytic at $s=0$.

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