A Proof of a Resolvent Estimate for Plane Couette Flow by New Analytical and Numerical Techniques

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Abstract

This thesis concerns stability of plane Couette flow in three space dimensions for the incompressible Navier-Stokes equations. We present new results for the resolvent corresponding to this flow. Previously, analytical bounds of the resolvent have been derived in parts of the unstable half-plane. In the remaining part, only bounds based on numerical computations in an infinite parameter domain are available. Due to the need for truncation of this infinite parameter domain, these results are mathematically insufficient.

We obtain a new analytical bound of the resolvent at $s = 0$ in all but a compact subset of the parameter domain. This is done by deriving approximate solutions of the Orr-Sommerfeldt equation and bounding the errors made by the approximations. In the remaining compact set, we use standard numerical techniques to obtain a bound. Hence, this part of the proof is not rigorous in the mathematical sense.

In the thesis, we present a way of making also the numerical part of the proof rigorous. By using analytical techniques, we reduce the remaining compact subset of the parameter domain to a finite set of parameter values. In this set, we need to compute bounds of the solution of a boundary value problem. By using a validated numerical method, such bounds can be obtained. In the last part of the thesis, we investigate a validated numerical method for enclosing the solutions of boundary value problems.

Preface

This thesis contains two papers and an introduction.

Paper I: Per-Olov Åsén and Gunilla Kreiss, *A Rigorous Resolvent Estimate for Plane Couette Flow*, (2004), submitted.

The author of this thesis contributed to the ideas, performed the numerical computations and wrote the manuscript.

Paper II: Malin Siklosi and Per-Olov Åsén, *On a Computer-Assisted Method for Proving Existence of Solutions of Boundary Value Problems* . Technical Report, NADA, Royal Institute of Technology, (2004).

The theoretical derivation were done in close cooperation between the authors, both of them contributing in an equal amount. The author of this thesis had the main responsibility for the computer implementations and wrote section 5 in the report. Malin Siklosi had the main responsibility for the literature studies and wrote sections 1–4 in the report.

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Contents

Chapter 1

Introduction

The topics of this thesis are stability of plane Couette flow and the use of computers to derive mathematical proofs. Plane Couette flow is the stationary flow of an incompressible fluid between two infinite parallel plates, moving in opposite directions at a constant speed. The mathematical model describing the flow is the Navier-Stokes equations. The reason for studying plane Couette flow is that it is a simple example of a shear flow and the analytical solution is known. A better understanding of the stability of plane Couette flow could provide information useful for more complicated flows.

Paper 1 concerns the stability of plane Couette flow by bounding the norm of the resolvent at $s = 0$. Previously derived bounds have been based on numerical computations in parts of an infinite parameter domain. We present new analytical results for the resolvent. These results imply a sharp bound of the resolvent in all but a compact subset of the infinite parameter domain. By reducing the domain where computations are needed to a compact set, it is possible to derive a mathematically rigorous bound by using a validated numerical method. This is the topic of paper 2, where we evaluate a method for proving existence and enclosures of solutions of boundary value problems by using numerical computations. Hence, paper 2 provides a way of making the entire bound of the resolvent in paper 1 rigorous.

The initial chapters in the thesis give a brief background to the topics of the two papers. In chapter 2, the Navier-Stokes equations are introduced, and some previous results are presented. The literature available on the Navier-Stokes equations is vast, and some references to books are given for further reading. In chapter 3, the use of computers for mathematical proofs is discussed. The idea of using computers for mathematically rigorous proofs is almost a contradiction. The inherent rounding errors in floating-point calculations and the necessity of finite dimensional models in a computer seems impossible to overcome. We give the basic ideas of how these obstacles can be conquered by using well known results from functional analysis and by using a different representation of real numbers when stored in a computer. The ideas are focused on the method implemented in paper 2. We also describe in detail how the method in paper 2 can be used to make the numerical part of the proof in paper 1 rigorous. Chapter 4 contains short summaries of the two papers in the thesis. The summaries are slightly more extensive than the corresponding abstracts and are included for the readers convenience.

Chapter 2

The Navier-Stokes Equations

A mathematical description of the flow of a viscous incompressible fluid was first derived in the early 19th century by Navier. Shortly after, others gave the equations a more firm mathematical foundation. The result was the widely known Navier-Stokes equations.

Given a domain $\Omega \subset \mathbb{R}^n$, let $\boldsymbol{u}(t, \boldsymbol{x}) = (u_1(t, \boldsymbol{x}), \ldots, u_n(t, \boldsymbol{x}))$ be the velocity and $p(t, \boldsymbol{x})$ the pressure at $(t, \boldsymbol{x}) = (t, x_1, \ldots, x_n)$. The nondimensionalized Navier-Stokes equations give the evolution of the flow as

$$
\begin{aligned}
\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p &= \frac{1}{R}\Delta \boldsymbol{u}, \\
\nabla \cdot \boldsymbol{u} &= 0.
\end{aligned}
\tag{2.1}
$$

Here, R is the Reynolds number given by $R = VL/\nu$, where V and L are typical velocity and length scales respectively and ν is the kinematic viscosity of the fluid. The equations must also be supplemented with initial and boundary conditions.

It is well known that in two space dimensions, (2.1) has a unique solution for all times under some restrictions on the initial condition. In three space dimensions, there are local (in time) existence results which can be extended to global existence results if the initial condition is small enough in some suitable norm, see e.g. [22] p. 345. Since the analytical solution of (2.1) is only known in a few special cases, deriving the solution usually involves the use of some numerical method. Existence and uniqueness results are then valuable. For further reading about the mathematical properties of the Navier-Stokes equations, we refer to [5], [22] and [28].

2.1 Hydrodynamic Stability

The field of hydrodynamic stability concerns the stability of various flows when subjected to perturbations. This is an important concept since a stationary unstable flow can not exist in reality. Also, a flow can be stable to some perturbations while unstable to others. A perturbation which grows with time might lead to turbulence. Quantifying for which perturbations a flow is stable is of great importance in various applications. For an introduction to the field, we refer to the books [2] and [19].

Given a flow $\boldsymbol{U}, P$ which solves (2.1), the effect of a perturbation, $\boldsymbol{u}^0$, can be investigated by considering the equations for the perturbed state. Let the deviation from the given flow be denoted by $\boldsymbol{u}, p$. Since both the given flow $\boldsymbol{U}, P$ and the perturbed state $\boldsymbol{U} + \boldsymbol{u}, P + p$

satisfy (2.1), subtracting the equations yields

$$
\begin{aligned}
\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{U} + \nabla \boldsymbol{p} &= \frac{1}{R}\Delta\boldsymbol{u}, \\
\nabla \cdot \boldsymbol{u} &= 0,
\end{aligned}
\tag{2.2}
$$

with initial condition $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}^0$.

In order to define stability, we need a norm to measure the size of the perturbation. The most commonly used norm is the L^2-norm, since it is equivalent to the kinetic energy of the perturbation. However, any norm can be used and in some cases other choices of norms might be more suitable.

The flow $\boldsymbol{U}$ is called stable to the perturbation $\boldsymbol{u}^0$ if the norm of the perturbation becomes arbitrarily small as time increases, i.e. if

$$
\lim_{t\to\infty} \|\boldsymbol{u}(t)\| = 0. \tag{2.3}
$$

If the flow is stable to all perturbations, it is called globally stable. Usually, the flow is only stable to all perturbations which are small enough, i.e. to all perturbations satisfying $\|\boldsymbol{u}^0\| < \gamma$ for some $\gamma > 0$. This is known as conditional stability.

The type of stability a flow exhibits typically depends on the Reynolds number, R. At low R, the flow might be globally stable, while being conditionally stable at higher R. Especially, some flows have a critical Reynolds number, R_C, such that for $R > R_C$, the flow is not conditionally stable. This means that there exists at least one infinitesimal perturbation such that the flow is not stable. Determining how the stability depends on the Reynolds number for different flows is of central interest in hydrodynamic stability.

2.2   Methods for Analyzing Stability

In this section, we present three ways to determine whether a flow is stable or not. We focus on describing the last technique, the use of the resolvent, since this relates to the topic of this thesis. There are, of course, other methods available as well, such as e.g. energy theory, and we refer to [2] and [19] for further reading.

The most straightforward way to get some information about the stability of a flow is to consider the eigenvalues of the linearized equations for the perturbation. If there is an eigenvalue with positive real part, perturbations with a non-zero component in the direction of the corresponding eigenfunction will exhibit exponential growth. Determining the smallest Reynolds number which allows exponentially growing perturbations gives the critical Reynolds number, R_C. However, this does not imply that for subcritical Reynolds numbers, i.e. for $R < R_C$, the flow is stable to all perturbations, since the effect of the nonlinear term is ignored. Hence, the eigenvalues give no information about the possible conditional stability at lower Reynolds numbers. Even for the linearized problem, the eigenvalues alone can sometimes give poor information of the short time behavior of a perturbation. If all eigenvalues are negative, the perturbations will eventually decay exponentially. However, there can be considerable growth initially. This is due to non-orthogonality of the eigenfunctions and is something we will return to shortly.

Detailed information about the stability of a flow can be obtained by using numerical simulations. For a given R, try an initial perturbation and solve the full nonlinear equations for a sufficiently long time, until it becomes clear if the perturbation vanishes or not. This strategy is quite new, since solving the full nonlinear problem, using direct numerical simulation, is computationally expensive. Also, rigorous results can never be obtained by

computations alone, since all possible perturbations can not be tested. Therefore, analytical results are necessary.

In order to analytically derive conditions for stability, the resolvent can be used. The resolvent is the solution operator of the Laplace transformed linearized problem. Assume that we have a bound of the norm of the resolvent in the entire unstable half-plane. Then it is possible to derive a bound of the solution of the forced linear problem. This bound is given in terms of the bound of the resolvent and the norm of the forcing. The linear bound is then extended to the nonlinear problem by treating the nonlinear term in the equation as part of the forcing. This is only possible if the forcing is sufficiently small. This condition gives a sufficient condition on the size of the perturbation under which nonlinear stability is guaranteed. We illustrate this method on a simple model problem, similar to the model problem treated in [4].

Example. Consider the following ordinary differential equation for $\boldsymbol{v} = (v_1, v_2)^T$,

$$
\begin{aligned}
\boldsymbol{v}_t &= L\boldsymbol{v} + \boldsymbol{g}(\boldsymbol{v}) + \boldsymbol{f}(t), \\
\boldsymbol{v}(0) &= \boldsymbol{v}^0,
\end{aligned}
\tag{2.4}
$$

where

$$
L = \begin{pmatrix} -R^{-1} & 0 \\ 1 & -2R^{-1} \end{pmatrix}, \quad \boldsymbol{g}(\boldsymbol{v}) = \begin{pmatrix} v_1 v_2 \\ v_1^2 \end{pmatrix}.
$$

We are interested in how the stability of this system changes when the positive constant R grows.

Consider first the linear, unforced case $\boldsymbol{g} = \boldsymbol{f} = 0$ with initial condition $\boldsymbol{v}^0 = (v_1^0, v_2^0)^T$. Since, with $R > 0$, the eigenvalues of L are negative, we know that the solution decays exponentially for sufficiently large times. However, the short time behavior can be significantly different. The general solution of this problem is given by

$$
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1^0 \begin{pmatrix} 1 \\ R \end{pmatrix} e^{-t/R} + (v_2^0 - v_1^0 R) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t/R}.
$$

We see that v_1 decays exponentially at all times. However, Taylor expanding v_2 at $t = 0$ shows that v_2 grows linearly for $t \leq O(R)$. This is known as transient growth, and is due to the fact that the operator L is non-normal, i.e. the eigenvectors of L are non-orthogonal. In fact, the eigenvectors of L are $(1, R)^T$ and $(0, 1)^T$, i.e. they are increasingly non-orthogonal with increasing R.

We now derive conditions for stability of the nonlinear problem. Since the resolvent method uses the Laplace transform, we consider (2.4) with homogeneous initial conditions. Note that (2.2) could be transformed to an equivalent homogeneous problem by e.g. introducing $\boldsymbol{u} = \boldsymbol{v} + e^{-\delta t}\boldsymbol{u}^0$ for some $\delta > 0$. This would result in a forcing involving the initial perturbation, $\boldsymbol{u}^0$, in the equations for $\boldsymbol{v}$.

Let $|\cdot|$ and $(\cdot, \cdot)$ denote the l^2-norm and inner product of vectors and let $\|\cdot\|$ denote the corresponding matrix norm. The linear problem corresponding to (2.4) is, after applying the Laplace transform,

$$
s\tilde{\boldsymbol{v}} = L\tilde{\boldsymbol{v}} + \tilde{\boldsymbol{f}}(s).
$$

The solution operator $(sI - L)^{-1}$ is known as the resolvent. With $R > 0$, the eigenvalues of L are negative. Hence, the resolvent is well defined in the entire unstable half-plane, $\mathrm{Re}(s) \geq 0$. For a normal operator, N, the norm of the resolvent, $\mathcal{R}(s) = (sI - N)^{-1}$, is given by $\|\mathcal{R}(s)\| = \sup_{\lambda \in \sigma(N)} |s - \lambda|^{-1}$, where $\sigma(N)$ is the spectrum of N, see e.g.

[3]. However, since L is non-normal, the norm of the resolvent is larger. Straightforward calculations give the sharp bound

$$\|(sI - L)^{-1}\| \leq CR^2$$

in the entire unstable half-plane.

We use this to bound the solution of the linear problem. By using Parseval's identity, it follows that

$$\int_0^T |\boldsymbol{v}(t)|^2 dt \leq \int_0^\infty |\boldsymbol{v}(t)|^2 dt \leq CR^4 \int_0^\infty |\boldsymbol{f}(t)|^2 dt.$$

For $t \leq T$, the solution $\boldsymbol{v}(t)$ does not depend on $\boldsymbol{f}(t)$ for $t > T$. Hence, we can set $\boldsymbol{f}(t) = 0$ for $t > T$ in the above expression, yielding

$$\int_0^T |\boldsymbol{v}(t)|^2 dt \leq CR^4 \int_0^T |\boldsymbol{f}(t)|^2 dt. \tag{2.5}$$

We also need a bound of $|\boldsymbol{v}(T)|$. Scalar multiplication of the linear equation corresponding to (2.4) with $\boldsymbol{v}$ gives

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{v}(t)|^2 = (\boldsymbol{v}, \boldsymbol{v}_t) = (\boldsymbol{v}, L\boldsymbol{v}) + (\boldsymbol{v}, \boldsymbol{f}) \leq C_1|\boldsymbol{v}|^2 + \frac{1}{2}(|\boldsymbol{v}|^2 + |\boldsymbol{f}|^2),$$

where C_1 is a bound of the range of L. Integrating this from $t = 0$ to $t = T$ and using (2.5) gives

$$|\boldsymbol{v}(T)|^2 \leq \tilde{C}R^4 \int_0^T |\boldsymbol{f}(t)|^2 dt.$$

Hence, we have the following bound for the linear problem,

$$|\boldsymbol{v}(T)|^2 + \int_0^T |\boldsymbol{v}(t)|^2 dt \leq C_L R^4 \int_0^T |\boldsymbol{f}(t)|^2 dt. \tag{2.6}$$

Now, we will treat the nonlinear term as part of the forcing. For the nonlinear term, we have

$$|\boldsymbol{g}(\boldsymbol{v})|^2 \leq |\boldsymbol{v}|^4. \tag{2.7}$$

Assume that the solution of the nonlinear problem (2.4) satisfies

$$\begin{aligned} |\boldsymbol{v}(T)|^2 &\leq 4R^4 K, \\ K &= C_L \int_0^\infty |\boldsymbol{f}(t)|^2 dt, \end{aligned} \tag{2.8}$$

for all times $T \in [0, \infty)$. We prove this assumption by assuming that it is not true, thus deriving a contradiction. Since $\boldsymbol{v}(0) = 0$, (2.8) must hold with strict inequality for some initial time interval. Let $T_0 > 0$ be the smallest time such that there is equality in (2.8) and consider $T \leq T_0$. From the linear estimate (2.6) and the bounds (2.7) and (2.8), we have

$$\begin{aligned} |\boldsymbol{v}(T)|^2 + \int_0^T |\boldsymbol{v}(t)|^2 dt &\leq C_L R^4 \int_0^T |\boldsymbol{f}(t) + \boldsymbol{g}(\boldsymbol{v})|^2 dt \leq 2C_L R^4 \int_0^T |\boldsymbol{f}(t)|^2 + |\boldsymbol{v}(t)|^4 dt \\ &\leq 2R^4 K + 8C_L R^8 K \int_0^T |\boldsymbol{v}(t)|^2 dt. \end{aligned}$$

We must now assume that the forcing is sufficiently small, which will give a condition for stability. Assume that

$$1 - 8C_L R^8 K \geq \frac{1}{2}. \tag{2.9}$$

Then for $T \leq T_0$, we have the following bound

$$|\boldsymbol{v}(T)|^2 + \frac{1}{2} \int_0^T |\boldsymbol{v}(t)|^2 dt \leq 2R^4 K. \tag{2.10}$$

Clearly, the assumption of equality in (2.8) at time T_0 can not be true and (2.10) must hold for all times $T \in [0, \infty)$.

If we assume $\boldsymbol{f}(t) \in L^2([0, \infty))$, the right hand side of (2.10) is bounded. It follows that $\boldsymbol{v}(t) \in L^2([0, \infty))$, which implies $\lim_{t \to \infty} |\boldsymbol{v}(t)| = 0$. Thus, we have proved nonlinear stability under the assumption (2.9), i.e. when

$$\int_0^\infty |\boldsymbol{f}(t)|^2 dt \leq \hat{C} R^{-8}.$$

2.3 Channel Flow

In this section we consider flow in a channel. There are two classical problems of this type; plane Couette flow and plane Poiseuille flow. Both flows involve an incompressible fluid between two infinite parallel plates and in both cases, the analytical solution is known. In plane Couette flow, the plates are moving in opposite directions at a constant speed and in plane Poiseuille flow, the plates are stationary and the flow is driven by a constant non-zero pressure gradient in the streamwise direction.

The coordinate system is chosen such that x_1 is the streamwise direction, x_2 the direction normal to the plates and x_3 the spanwise direction. The plates are located at $x_2 = \pm 1$, i.e. the domain is $\Omega = \{\boldsymbol{x} \in \mathbb{R}^3 : -1 < x_2 < 1\}$. In both cases, the stationary flow is parallel and given by

$$U = (U_1(x_2), 0, 0)$$

where $U_1(x_2)$ is

$$U_1(x_2) = \begin{cases} x_2, & \text{for plane Couette flow,} \\ 1 - x_2^2, & \text{for plane Poiseuille flow.} \end{cases}$$

The flows are shown in Figure 2.1.

Plane Couette flow and plane Poiseuille flow have been extensively studied by applied mathematicians throughout the years, mainly because they are some of the simplest examples of flows available. However, despite their simplicity much is still unknown about the effects of perturbations on the stationary flows. A better understanding of the important mechanisms in these flows could have implications for other, more complicated, flows.

Romanov proved in 1973, [18], that plane Couette flow is linearly stable at all Reynolds numbers. In experiments however, turbulence has been observed at Reynolds numbers as low as $R \approx 350$. Plane Poiseuille flow is linearly unstable for $R > 5772.22$, [12], when the so called Tollmien-Schlichting wave becomes linearly unstable. Also here, turbulence typically appears at much lower R in reality. By direct numerical simulations, the threshold amplitude below which perturbations eventually decay has been examined. In e.g. [8], the thresholds were found to behave as $R^{-\beta}$, with $\beta \approx 1.25$ for plane Couette flow and $\beta \approx 1.75$ for plane Poiseuille flow. Also, for plane Poiseuille flow, the perturbation which requires the smallest amplitude for transition to turbulence at subcritical Reynolds numbers is not
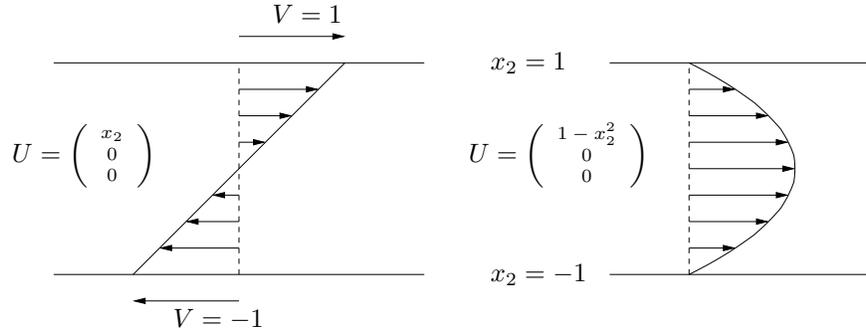
Figure 2.1: Stationary solution of plane Couette flow (left) and plane Poiseuille flow (right)

the Tollmien-Schlichting wave, [16]. Hence, the spectrum gives poor information about the influence of perturbations in these flows.

It was found in [17] that the eigenfunctions of the linearized operators of plane Couette flow and plane Poiseuille flow are highly non-normal in the L^2-inner product. This can cause significant transient growth, as explained in the previous section. Therefore, the resolvent or the ε-pseudospectrum has been in focus the last decade. The ε-pseudospectrum of a linear operator, L, generalizes the concept of eigenvalues by defining s to belong to the ε-pseudospectrum if $\|(sI - L)^{-1}\| \geq \varepsilon^{-1}$. Hence, the ε-pseudospectrum gives information of where the resolvent is large, as opposed to the spectrum which only give information of where the resolvent is infinite or non-existing. Computations of the ε-pseudospectrum for plane Couette flow and plane Poiseuille flow can be found in [24].

The resolvent, $\mathcal{R}$, for plane Couette flow, has been investigated by numerical and analytical techniques. In [4], the lower bound $\|\mathcal{R}\| \geq CR^2$ was proved for the L^2-norm. Numerical computations in [4] and [7] indicated this asymptotic dependence to hold in the entire unstable half-plane, i.e. that $\|\mathcal{R}\| \leq CR^2$. An analytical bound of the L^2-norm of the resolvent was derived in large parts of the unstable half-plane in [7], where also a new norm was introduced. Computations in [7] indicated the optimal R-dependence $\|\mathcal{R}\| \leq CR$ in the new norm, which means that the linearized operator is normal in the scalar product corresponding to the new norm.

As in the example in the previous section, a bound on the resolvent can be used to prove nonlinear stability. Using this technique, the upper bound $\beta \leq 5.25$ in the threshold amplitude dependence $R^{-\beta}$ was proved in [4]. Since the resolvent bounds available are based on numerical computations in an infinite parameter domain, the proof is not fully rigorous. This is the motivation of the first paper of this thesis. We present a new sharp bound of the resolvent at the believed maximum $s = 0$. The bound is based on analytical estimates in all but a compact subset of the parameter domain. In this compact set, we use numerical computations to obtain a bound. Since the set is compact, the numerical bound can be made rigorous by using validated numerical methods. We explain in detail how this can be done in the next chapter. Using the same technique, we hope to bound the resolvent in the remaining part of the unstable half-plane in the future. Moreover, analytical bounds can provide more precise information about the resolvent than just the maximum in the unstable half-plane. Such information could be used to improve the upper bound of β, i.e. to sharpen the threshold amplitude for nonlinear stability.

Chapter 3

Computer-Assisted Proofs

The invention of the computer has had a tremendous impact on the field of applied mathematics. Problems that were practically impossible to solve 50 years ago are solved in fractions of a second today. However, these solutions are almost never true solutions. A numerical solution of a problem usually suffers from errors. One source of error is that the mathematical model might have infinite degrees of freedom, making finite dimensional approximations necessary. Deriving explicit bounds on the errors made by the approximations is usually difficult. Another source is the rounding error. Numbers like π, $\sqrt{2}$ can not be stored exactly in a computer. Even for numbers that are stored exactly, floating-point arithmetic is not closed. This means that even if x and y can be stored exactly, there is no guarantee that e.g. $x + y$ can be stored exactly, making rounding necessary.

In this chapter, we give the basic ideas of how to prove existence and enclosures of solutions of elliptic boundary value problems. This is the topic of the second paper in this thesis. We also explain why this topic is relevant for the first paper of this thesis.

3.1 Basic Ideas

In this section, we describe two methods for proving existence of solutions of elliptic boundary value problems. The first method was proposed by Nakao, and has been successfully used in various applications, [10, 23, 25, 26, 27]. This is the method used in paper 2 of this thesis. The second method was proposed by Plum, and has also proved successful, [1, 6, 13, 14, 15]. The methods are quite similar in some parts, and a combination of them has been used by Nagatou, Yamamoto and Nakao, [11].

Both methods rely on an approximate, numerical solution, u_h, which can be derived by any numerical method. From the approximate solution, a suitable fixed-point equation, $w = T(w)$, for the error, $w = u - u_h$, is derived. The idea is to prove that $w = T(w)$ has a solution in a subset of a Banach space. The subset consists of functions with norm smaller than an explicitly derived upper bound. This upper bound gives bounds of the magnitude of the error in the approximate solution, u_h.

In order to prove the existence of a solution of the fixed-point equation, Nakao and Plum use the well known Schauder fixed-point theorem or Banach fixed-point theorem. The theorems state, see e.g. [29],

Theorem 3.1.1 (Schauder fixed-point theorem). *Let W be a non-empty, closed, bounded, convex subset of a Banach space X. If $T : W \to W$ is a compact operator, then there exists a $w \in W$ such that $w = T(w)$.*

Theorem 3.1.2 (Banach fixed-point theorem). *Let W be a non-empty, closed subset of a complete metric space X. If $T : W \rightarrow W$ is a contraction on W, then there exists a unique $w \in W$ such that $w = T(w)$.*

Note that Theorem (3.1.2) ensures a unique solution in W, which is not the case for Theorem (3.1.1).

Verifying that Theorem (3.1.1) or Theorem (3.1.2) can be applied to the derived fixed-point equation and finding a suitable subset are done in different ways in the approaches by Nakao and Plum.

In Nakao's method, the fixed-point equation is divided into a finite dimensional part and an infinite dimensional part. The finite dimensional part is rewritten using the linearization, L_h, of the finite dimensional projection of the given equation at the approximate solution, u_h. This yields an equivalent fixed-point equation which is more likely to map the finite dimensional part of W into itself. Verifying the conditions of Theorem (3.1.1) or Theorem (3.1.2) for the finite dimensional part is done by explicitly inverting L_h. The infinite dimensional part is treated by analytical methods, using e.g. a priori error bounds on the projection into the finite dimensional subspace.

Plum's method uses the linearization, L, of the infinite dimensional problem at the approximate solution, u_h. Using a lower bound of the norm of L and a bound of the norm of the residual of u_h, the conditions of Theorem (3.1.1) or Theorem (3.1.2) are verified by analytical and numerical techniques. The main difficulty is to derive the lower bound of the norm of L. This is obtained from the eigenvalue of L or L^*L with smallest absolute value. Deriving an enclosure of this eigenvalue can be done by solving related finite-dimensional matrix eigenvalue problems which is suitable for computer implementation.

In both methods, the effect of the rounding errors in computations must be accounted for. This can be done by using interval arithmetic, [9]. Interval arithmetic represents real numbers as closed intervals, where the upper and lower bounds of the intervals are floating-point numbers. Thus, all real numbers can be represented. By defining an arithmetic for the intervals, the effect of the rounding error can be rigorously accounted for in each arithmetic operation. This can be extended to all elementary functions used in computations, such that the functions take intervals as arguments and return intervals which encloses the range of the functions over the argument intervals.

3.2   Relation to Paper 1

In the first paper of this thesis, a bound of the resolvent for plane Couette flow is derived at the point $s = 0$. This is done by obtaining analytical bounds in all but a compact subset of an infinite parameter domain consisting of wave numbers in two space directions and the Reynolds number. In the remaining compact set, we use standard numerical computations for a finite set of these parameter values. However, although the subset of the parameter domain is bounded, it consists of infinitely many parameter values. Thus, this part of the proof is not rigorous. In this section, we describe how the method in paper 2 could be used to make also the numerical part of the proof rigorous.

In paper 1, the numerical part of the proof concerns the boundary value problem

$$
\begin{aligned}
w''(x) - (iax + b^2)w(x) &= 0, \\
w(-1) &= 1, \\
w(1) &= 0,
\end{aligned}
\tag{3.1}
$$

in the compact parameter domain $\Sigma = \{a, b \in \mathbb{R} \mid a \in [1/16, 40^3], \ b^2 \in [0, a^{2/3}]\}$. For every combination of a and b in Σ, we need to prove two things about the solution of (3.1). First,

we need to prove that the L^2-norm of the solution is bounded. Later in this section, we show that this holds for all parameter values in Σ, see the remark after the proof of Lemma 3.2.1. Secondly, we need to prove that the matrix

$$J = \begin{pmatrix} u'(-1) & -(u^*)'(1) \\ u'(1) & -(u^*)'(-1) \end{pmatrix} \tag{3.2}$$

is non-singular. Here, $(u^*)(x)$ denotes the complex conjugate of $u(x)$. The matrix elements are given by

$$u'(-1) \;\; = \;\; \int_{-1}^{1} f_b(\sigma) w(\sigma) d\sigma, \tag{3.3}$$

$$u'(1) \;\; = \;\; \int_{-1}^{1} g_b(\sigma) w(\sigma) d\sigma, \tag{3.4}$$

where

$$f_b(\sigma) \;\; = \;\; \begin{cases} \frac{\sinh(b(\sigma-1))}{\sinh(2b)}, & b \neq 0, \\ \frac{\sigma-1}{2}, & b = 0, \end{cases} \tag{3.5}$$

$$g_b(\sigma) \;\; = \;\; \begin{cases} \frac{\sinh(b(\sigma+1))}{\sinh(2b)}, & b \neq 0, \\ \frac{\sigma+1}{2}, & b = 0. \end{cases} \tag{3.6}$$

Note that the matrix (3.2) is non-singular if and only if $|u'(-1)| \neq |u'(1)|$. Hence, for a given pair of parameters, a and b, we need to enclose the solution $w(x)$ of (3.1) and then derive rigorous enclosures of the absolute values of the integrals (3.3) and (3.4). This can be done with the method used in the second paper of this thesis. However, since we implemented the method in MATLAB, we were not able to obtain rigorous bounds when a is large. Using e.g. Fortran would hopefully be sufficient for covering the parameter domain we are interested in.

We are still left with the problem of having an infinite number of parameter values in Σ. This can be handled with analytical techniques. By using information about how far from singular J is at a given point in Σ, we are able to prove that J is non-singular in a neighborhood around this point. The result is summarized in the following lemma.

Lemma 3.2.1. *If for $a = A$ and $b = B$, the solution $W(x)$ of (3.1) is such that the matrix elements (3.3) and (3.4) satisfy*

$$|U'(-1)| - |U'(1)| \geq \alpha \tag{3.7}$$

for some $\alpha > 0$. Then the matrix J given by (3.2) is non-singular for all parameter values a and b satisfying

$$8\beta(\|f_B\| + \|g_B\|) + (8\beta + 1)(\|f_b - f_B\| + \|g_b - g_B\|) < \frac{\alpha}{\|W\|}, \tag{3.8}$$

where f and g are given by (3.5) and (3.6) and where

$$\beta = |a - A| + |b^2 - B^2|.$$

Here, $\| \cdot \|$ is the L^2-norm on $\Omega = \{-1 \leq x \leq 1\}$.

Before proving the lemma, note that it is not obvious that the quantity on the left hand side of (3.7) should be positive. However, numerical experiments indicate this to always be the case. Of course, if the left hand side of (3.7) would be negative for some parameter combination, a similar lemma could be derived handling this case.

Proof. Consider some parameter values a and b satisfying (3.8) and denote the corresponding solution of (3.1) by $w(x)$. From (3.1), the difference $\tilde{w} = w - W$ satisfies

$$\tilde{w}'' - (iax + b^2)\tilde{w} = (i(a - A)x + (b^2 - B^2))W, \quad \tilde{w}(\pm 1) = 0.$$

Taking the L^2-inner product of this equation with $\tilde{w}$, using integration by parts and taking the real part yields

$$\|\tilde{w}'\|^2 + b^2\|\tilde{w}\|^2 \le (|a - A| + |b^2 - B^2|)\|W\|\|\tilde{w}\| = \beta\|W\|\|\tilde{w}\|,$$

Using a Poincaré inequality for $\tilde{w}$ and the relation $cd \le c^2/(2\mu) + d^2\mu/2$, valid for all $c, d \in \mathbb{R}$, $\mu > 0$, we thus have the bound

$$\frac{1}{2}\|\tilde{w}'\|^2 + \left(\frac{1}{16} + b^2\right)\|\tilde{w}\|^2 \le 4\beta^2\|W\|^2. \tag{3.9}$$

Now, evaluating $|u'(-1)|$ from (3.3), using $w = \tilde{w} + W$ and (3.9) gives

$$
\begin{aligned}
|u'(-1)| &= \left|\int_{-1}^{1} (f_B(\sigma) + f_b(\sigma) - f_B(\sigma))(\tilde{w}(\sigma) + W(\sigma))d\sigma\right| \\
&\ge |U'(-1)| - \|f_B\|\|\tilde{w}\| - \|f_b - f_B\|(\|\tilde{w}\| + \|W\|) \\
&\ge |U'(-1)| - 8\beta\|f_B\|\|W\| - (8\beta + 1)\|f_b - f_B\|\|W\|.
\end{aligned} \tag{3.10}
$$

Similarly, using (3.4) yields

$$|u'(1)| \le |U'(1)| + 8\beta\|g_B\|\|W\| + (8\beta + 1)\|g_b - g_B\|\|W\|. \tag{3.11}$$

By (3.7), (3.8), (3.10), and (3.11), we have $|u'(-1)| - |u'(1)| > 0$ and thus J is non-singular. □

Remark. We stated earlier in this section that the L^2-norm of the solution of (3.1) is bounded for all a and b in Σ. Since (3.9) also holds when W is the solution with A and B outside Σ, we can especially chose $A = B = 0$. Clearly, $\|W\|$ is then bounded, and it follows from (3.9) that $\|w\| = \|\tilde{w} + W\|$ is bounded in any bounded parameter domain.

Hence, Lemma 3.2.1 and the method used in paper 2 provides a possibility of deriving a rigorous bound of the resolvent in Σ, where the bound in paper 1 is not rigorous. One needs to find a finite set of points in Σ such that J is non-singular for these points and the neighborhoods, given by Lemma 3.2.1, cover Σ.

In order for Σ to be covered, we must ensure that the measures of the neighborhoods do not become arbitrarily small even if J is non-singular. This can only happen if α in (3.7) becomes arbitrarily small somewhere in Σ. However, from (3.10) and (3.11), we know that the function $\gamma(a, b) \equiv |u'(-1)| - |u'(1)|$ is continuous with respect to a and b. Since Σ is a compact set, $\gamma(a, b)$ attains a minimum, $\alpha_{\min}$, in Σ. Hence, if J is non-singular in Σ, we can cover Σ with a finite number of neighborhoods attained from using Lemma 3.2.1. Computations made in paper 1 indicate that J is non-singular, and we believe this could be proved with the approach described in this section.

Finally, note that when computing the quantities in (3.8), all computations should be rigorous, using e.g. interval arithmetic. Since $\|f_B\|$, $\|g_B\|$, $\|f_b - f_B\|$ and $\|g_b - g_B\|$ can be derived explicitly, implementation using interval arithmetic is straightforward.

Chapter 4

Summary of Papers

4.1 Paper 1: A Rigorous Resolvent Estimate for Plane Couette Flow

In this paper, we derive a rigorous bound of the resolvent for plane Couette flow at the point $s = 0$. We do this analytically by finding approximate solutions of the Orr-Sommerfeldt equation while keeping track of the errors made by the approximations. This is not possible in the entire parameter domain. However, the remaining domain is bounded, and we use numerical computations to obtain a bound. Previously derived bounds at $s = 0$ have been based on computations in an infinite parameter domain, making rigorous results impossible. In a bounded domain, rigorous results can be derived by the use of numerical verification methods using interval arithmetic.

This paper is submitted to Journal of Mathematical Fluid Mechanics and is entry [20] in the bibliography.

4.2 Paper 2: On a Computer-Assisted Method for Proving Existence of Solutions of Boundary Value Problems

In paper 2, we investigate a method for proving existence of solutions of elliptic boundary value problems. The method was proposed by Nakao. We solve two problems using this method; a linear test problem and the one-dimensional viscous Burgers' equation. For the first problem, the method works well. For Burgers' equation however, the computational complexity becomes too large when the viscosity decreases. This is not surprisingly, since Burgers' equation linearized at the correct solution rapidly becomes close to singular when the viscosity is decreased. We therefore reformulate the problem by replacing one of the boundary condition with a global integral condition. This approach drastically reduces the computational complexity.

This paper is a technical report and is entry [21] in the bibliography.

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